Miscellaneous Exercise

Find the value of the following :

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

SOLUTION

$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6}$ as the range of principal value branch of $\cos^{-1}$ is $[0,\ \pi]$

So, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

SOLUTION

$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$ as the range of principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

So, $\tan^{=1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi+\frac{\pi}{6}\right)\right\} = \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

SOLUTION

Let $\sin^{-1}\frac{3}{5} = x \Rightarrow \frac{3}{5} = \sin x \Rightarrow \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

$\Rightarrow 2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{1-\frac{9}{16}}\right) = \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

SOLUTION

Let $\sin^{-1}\frac{8}{17} = x \Rightarrow \sin x = \frac{8}{17} \Rightarrow \tan x - \frac{8}{15}$ Let $\sin^{-1}\frac{3}{5} = y \Rightarrow \frac{3}{5} = \sin y \Rightarrow \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1}\frac{3}{4}$

$\Rightarrow x+y = \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$

$= \tan^{-1}\left(\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right) = \tan^{-1}\left(\frac{\frac{32+45}{60}}{1-\frac{24}{60}}\right) = \tan^{-1}\left(\frac{77}{36}\right)$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = cos^{-1}\frac{33}{65}$

SOLUTION

Let $x=\cos^{-1}\frac{4}{5}$ and $y=\cos^{-1}\frac{12}{13} \Rightarrow \cos x=\frac{4}{5}$ *and* $\cos y=\frac{12}{13}$

Now, $\sin x=\sqrt{1-\cos^2 x}$ *and* $\sin y=\sqrt{1-\cos^2 y}$

$\Rightarrow \sin x=\sqrt{1-\frac{16}{25}}$ *and* $\sin y=\sqrt{1-\frac{144}{169}} \Rightarrow \sin x=\frac{3}{5}$ *and* $\sin y=\frac{5}{13}$

We know that, $\cos(x+y)=\cos x\cos y-\sin x\sin y=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}$

$\Rightarrow \cos(x+y)=\frac{48}{65}-\frac{15}{65}=\frac{33}{65} \Rightarrow x+y=\cos^{-1}\left(\frac{33}{65}\right)$

Or, $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\left(\frac{33}{65}\right)$

6. $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

SOLUTION

Let $x=\cos^{-1}\frac{12}{13}$ *and* $y=\sin^{-1}\frac{3}{5}$ Or $\cos x=\frac{12}{13}$ *and* $\sin y=\frac{3}{5}$

Now, $\sin x=\sqrt{1-\cos^2 x}$ *and* $\cos y=\sqrt{1-\sin^2 y}$

$\Rightarrow \sin x=\sqrt{1-\frac{144}{169}}$ *and* $\cos y=\sqrt{1-\frac{9}{25}} \Rightarrow \sin x=\frac{5}{13}$ *and* $\cos y=\frac{4}{5}$

We know that, $\sin(x+y)=\sin x\cos y+\cos x\sin y=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20}{65}+\frac{36}{65}=\frac{56}{65}$

$\Rightarrow x+y=\sin^{-1}\left(\frac{56}{65}\right)$ *or,* $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

7. $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

SOLUTION

Let $\sin^{-1}\frac{5}{13}=x$ *and* $\cos^{-1}\frac{3}{5}=y \Rightarrow \sin x=\frac{5}{13}$ *and* $\cos y=\frac{3}{5}$ *or,* $\tan x=\frac{5}{12}$ *and* $\tan y=\frac{4}{3}$

$\Rightarrow x+y=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)=\tan^{-1}\left(\frac{\frac{5+16}{12}}{\frac{4}{9}}\right)$

$=\tan^{-1}\left(\frac{21}{12}\times\frac{9}{4}\right)=\tan^{-1}\frac{63}{16}$

8. $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

SOLUTION

L.H.S.$=\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}\right)+\left(\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}\right)+\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right)=\tan^{-1}\left(\frac{\frac{8}{15}}{\frac{14}{15}}\right)+\tan^{-1}\left(\frac{\frac{15}{56}}{\frac{55}{56}}\right)$

$=\tan^{-1}\frac{8}{14}+\tan^{-1}\frac{15}{55}=\tan^{-1}\frac{3}{11}$

$=\tan^{-1}\left[\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right]=\tan^{-1}\left(\frac{\frac{65}{77}}{\frac{65}{77}}\right)=\tan^{-1}1=\frac{\pi}{4}=R.H.S.$

Prove that

9. $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x\in[0,\ 1]$

SOLUTION

Putting $x=\tan^2\theta$, we get R.H.S.$=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\frac{1}{2}\times 2\theta=\theta=\tan^{-1}\sqrt{x}=R.H.S.$

$\therefore \tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\ x\in\left(0,\ \frac{\pi}{4}\right)$

SOLUTION

L.H.S. $=\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right\}$

$=\cot^{-1}\left\{\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}\right\}=\cot^{-1}\left\{\frac{2(1+\cos x)}{2\sin x}\right\}=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$=\cot^{-1}\left\{\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\}=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=R.H.S.$

11. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le 1$

[Hint : Put $x=\cos 2\theta$]

SOLUTION

Putting $x=\cos\theta$, *we get* L.H.S. $=\tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$

$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2(\theta/2)}-\sqrt{2\sin^2(\theta/2)}}{\sqrt{2\cos^2(\theta/2)}+\sqrt{2\sin^2(\theta/2)}}\right\}$ [Dividing numerator and denominator by cos

$(\theta/2)]=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}=\frac{\pi}{4}-\frac{\theta}{2}=\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)=R.H.S.$

Hence, L.H.S. = R.H.S.

12. $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

SOLUTION

$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\left[\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right]=\frac{9}{4}\cos^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Hence, L.H.S. = R.H.S.

Solve the following equations :

13. $2\tan^{-1}(\cos x)=\tan^{-1}(2\mathrm{cosec}x)$

SOLUTION

We have, $2\tan^{-1}(\cos x)=\tan^{-1}(2\mathrm{cosec}x)\Rightarrow\tan^{-1}\left[\frac{2\cos x}{1-\cos^2 x}\right]=\tan^{-1}(2\mathrm{cosec}x)$

$\Rightarrow\tan\left[\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)\right]=2\ \mathrm{cosec}x$

$\Rightarrow\frac{2\cos x}{\sin^2 x}=2\mathrm{cosec}x\Rightarrow\cos x=\sin x$

$\Rightarrow\tan x=1\ \ \Rightarrow x=\tan^{-1}1\Rightarrow x=\frac{\pi}{4}$

14. $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ (x>0)$

SOLUTION

We have, $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ (x>0)$

$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \Rightarrow \frac{3}{2}\tan^{-1}x=\tan^{-1}1=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{4}\times\frac{2}{3}=\frac{\pi}{6}\Rightarrow x=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow x=\frac{1}{\sqrt{3}}$

15. $\sin(\tan^{-1}x),\ |x|\ <1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$

(B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$

(D) $\frac{x}{\sqrt{1+x^2}}$

SOLUTION

(D) Let $\tan^{-1}x=\theta\Rightarrow x=\tan\theta,\ where\ \ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\therefore \sin(\tan^{-1}x)=\sin\theta$

Now, $\sin\theta=\frac{1}{\text{cosec}\theta}=\frac{1}{\sqrt{1+\cot^2\theta}}=\frac{1}{\sqrt{1+\frac{1}{\tan^2\theta}}}==\frac{x}{\sqrt{x^2+1}}$

16. . If $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2},\ then\ \ x$ is equal to

(A) 0, $\frac{1}{2}$

(B) 1, $\frac{1}{2}$

(C) 0

(D) $\frac{1}{2}$

SOLUTION

(C) $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}\Rightarrow\sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x\Rightarrow(1-x)=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)=\cos(2\sin^{-1}x)\Rightarrow 1-x=\cos(\cos^{-1}(1-2x^2)\Rightarrow 1-x=1-2x^2\Rightarrow 2x^2-x=0\Rightarrow x=0,\ \frac{1}{2}$ But, $x=\frac{1}{2}$ does not satisfy the equation. So, x = 0.

17. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $\frac{-3\pi}{4}$

SOLUTION

(C) $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\tan^{-1}\left(\frac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right)$

$=\tan^{-1}\left[\frac{x(x+Y)-y(x-y)}{y(x+y)+x(x-y)}\right]=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)=\tan^{-1}1=\frac{\pi}{4}.$

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