NCERT - Exercise 2.2

Prove the Following :

1. $3\sin^{-1}x = \sin^{-1}(3x-4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

SOLUTION

Put $\sin^{-1}x = \theta.\ Then,\ x = \sin\theta$ Now, $\sin 3\theta = (3\sin\theta - 4\sin^3\theta) = (3x-4x^3)$
$\Rightarrow 3\theta = \sin^{-1}(3x-4x^3)$
$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x-4x^3)$ Hence, $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

2. $3\cos^{-1}x = \cos^{-1}(4x^3-3x),\ x \in \left[\frac{1}{2}, 1\right]$

SOLUTION

Put $\cos^{-1}x = \theta.\ Then,\ x = \cos\theta$
Now, $\cos 3\theta =)4\cos^3\theta - 3\cos\theta) = (4x^3-3x) \Rightarrow 3\theta = \cos^{-1}(4x^3-3x) \Rightarrow 3\cos^{-1}x = \cos^{-1}(4x^3-3x)$
Hence, $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

SOLUTION

$$\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left[\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{18+77}{264}}{\frac{264-14}{264}}\right] = \tan^{-1}\left[\frac{125}{250}\right] = \tan^{-1}\left(\frac{1}{2}\right)$$

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

SOLUTION

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\frac{(2\times\frac{1}{2})}{\left\{1-\left(\frac{1}{2}\right)^2\right\}} + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{3/4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right] = \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right) = \tan^{-1}\left(\frac{31}{17}\right)$$

$\therefore 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Write the following functions in the simplest form :

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$

SOLUTION

Putting $x = \tan\theta$, we get $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left[\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right] = \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x$$

6. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}},\ |x| > 1$

SOLUTION

Putting $x = \sec\theta$, *we get* $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right) = \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\}$

$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1}x \therefore \tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \frac{\pi}{2} - \sec^{-1}x$

7. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right),\ x < \pi$

SOLUTION

$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

8. $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right),\ 0 < x < \pi$

SOLUTION

$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$ (Dividing numerator and denominator by cos x) $= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-x\right)\right) = \frac{\pi}{4} - x$

9. $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right),\ |x| < a$

SOLUTION

Put x = $asin\theta$ we get :

$tan^{-1}(\frac{asin\theta}{acos\theta})$

$= tan^{-1}(tan\theta) = \theta$

$= sin^{-1}(\frac{x}{a})$

10. $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right),\ a > 0;\ \dfrac{-a}{\sqrt{3}} \le x \le \dfrac{a}{\sqrt{3}}$.

SOLUTION

Putting $x = a\ \tan\theta$, we get $\tan^{-1}\left[\frac{3a^2x - x^3}{a^3 - 3ax^2}\right] = \tan^{-1}\left[\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right]$

$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$

Find the values of each of the following :

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

SOLUTION

We have $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)\right\} = \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\}$

$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\} = \tan^{-1}\left[2\times\frac{1}{2}\right] = \tan^{-1}1 = \frac{\pi}{4}$

12. $\cot(\tan^{-1}a+\cot^{-1}a)$

SOLUTION

$\cot(\tan^{-1}a+\cot^{-1}a)=\cot\left(\tan^{-1}a+\frac{\pi}{2}-\tan^{-1}a\right)=\cot\frac{\pi}{2}=0$

13. $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right],\ |x|\ <1,\ y>0$ and $xy<1$

SOLUTION

Putting $x=\tan\theta$ and $y=\tan\phi$, we get $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}=\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\right.$

$=\tan\left\{\frac{1}{2}\sin^{-1}(\sin 2\theta)+\frac{1}{2}\cos^{-1}(\cos 2\phi)\right\}$

$=\tan\left\{\frac{1}{2}\times 2\theta+\frac{1}{2}\times 2\phi\right\}=\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{x+y}{1-xy}$

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x

SOLUTION

$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$ Or $\sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1\Rightarrow\sin^{-1}\frac{1}{5}+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{2}\Rightarrow\sin^{-1}\frac{1}{5}=\sin^{-1}x$

$\Rightarrow x=\sin\left(\sin^{-1}\frac{1}{5}\right)=\frac{1}{5}.$

15. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x

SOLUTION

$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}\Rightarrow\tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$

$\Rightarrow\tan^{-1}\left(\frac{(x-1)(x+2)+(x+1)(x-2)}{x^2-4-(x^2-1)}\right)=\frac{\pi}{4}$

$\Rightarrow\frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-x^2+1}=\tan\frac{\pi}{4}$

$\Rightarrow\frac{2x^2-4}{-3}=1\Rightarrow 2x^2=-3+4\Rightarrow x^2=\frac{1}{2}\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

Find the value of each of the expressions :

16. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

SOLUTION

$\sin^{1}\left(\sin\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$ as the principal value branch of $\sin^{-1}$ *is* $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$

So, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)$

$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$

Hence, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\frac{\pi}{3}$

17. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

SOLUTION

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$ as the principal value branch of $\tan^{-1}$ *is* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{4}\right)\right]$

$= \tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

$= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$.

18. $\tan\left[\left(\sin^{-1}\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right]$

SOLUTION

Let $A = \sin^{-1}\frac{3}{5}$ and $B = \cot^{-1}\frac{3}{2} \Rightarrow \sin A = \frac{3}{5}$ *and* $\cot B = \frac{3}{2} \Rightarrow \tan A = \frac{3}{4}$ *and* $\tan B = \frac{2}{3}$

So, $\tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) = \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right) = \tan\left[\tan^{-1}\left(\frac{\frac{9+18}{12}}{1-\frac{1}{2}}\right)\right]$

$= \tan\left(\tan^{-1}\left(\frac{17}{16}\right)\right) = \frac{17}{6}$

19. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$

(B) $\frac{5\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{6}$

SOLUTION

(B) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$ as principal value branch of $\cos^{-1}$ *is* $[0, \pi]$

So, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(\pi + \frac{\pi}{6}\right)\right]$

$= \cos^{-1}\left(-\cos\frac{\pi}{6}\right) = \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{6}\right)\right) = \frac{5\pi}{6}$.

Hence, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$.

20. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to (A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) 1

SOLUTION

(D) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

(A) π

(B) $-\dfrac{\pi}{2}$

(C) 0

(D) $2\sqrt{3}$

SOLUTION

(B) $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3})$

$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3} = \dfrac{\pi}{3} - \pi + \dfrac{\pi}{6} = \dfrac{\pi}{2} - \pi = -\dfrac{\pi}{2}.$

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