NCERT - Miscellaneous Exercise

Integrate the functions in Exercises 1. to 24. :

1. $\dfrac{1}{x-x^3}$

SOLUTION

Let $I=\int \frac{dx}{x-x^3}$ and Let $\frac{1}{x-x^3}=\frac{1}{x(x+1)(1-x)}=\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x} \Rightarrow 1=A(1+x)(1-x)+Bx(1-x)+Cx(1+x)$ ....(i) Putting $x=0$ in (i), we get $1=A(1+0)(1-0) \Rightarrow A=1$ Putting $x=-1$ in (i), we get $1=B(-1)(1+1) \Rightarrow B=-\frac{1}{2}$ Putting $x=1$ in (i), we get $1=C(1)(1+1) \Rightarrow C=\frac{1}{2} \therefore \frac{1}{x-x^3}=\frac{1}{x}-\frac{1}{2(1+x)}+\frac{1}{2(1-x)} \Rightarrow I=\int \frac{1}{x-x^3}dx=\int \frac{1}{x}dx-\frac{1}{2}\int \frac{1}{1+x}dx+\frac{1}{2}\int \frac{1}{1-x}dx=\log|x|-\frac{1}{2}\log|1+x|-\frac{1}{2}\log|1-x|+C=\log|x|-\frac{1}{2}\log\left|1-x^2\right|+C=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$

2. $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$

SOLUTION

Let $I=\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx=\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}dx=\int \frac{\sqrt{x+a}-\sqrt{x+b}}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+b}\right)^2}dx\frac{1}{a-b}\int \left[(x+a)^{1/2}-(x+b)^{1/2}\right]$

$$=\frac{1}{a-b}\left[\frac{(x+a)^{3/2}}{\frac{3}{2}}-\frac{(x+b)^{3/2}}{\frac{3}{2}}\right]+C=\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+C$$

3. $\dfrac{1}{x\sqrt{ax-x^2}}$ [Hint : Put $x=\frac{a}{t}$]

SOLUTION

Let $I=\int \frac{1}{x\sqrt{ax-x^2}}dx$ Put $x=\frac{a}{t} \Rightarrow dx=-\frac{a}{t^2}dt$ Now, $x\sqrt{ax-x^2}=\frac{a}{t}\sqrt{\frac{a^2}{t}-\frac{a^2}{t^2}}=\frac{a^2}{t}\sqrt{\frac{1}{t}-\frac{1}{t^2}}=\frac{a^2}{t^2}\sqrt{t-1} \therefore I=\int \frac{1}{\frac{a^2}{t^2}\sqrt{t-1}}\times\left(-\frac{a}{t^2}\right)dt=-\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt=-\frac{1}{a}\cdot\frac{(t-1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=-\frac{2}{a}\sqrt{t-1}+C=-\frac{2}{a}\sqrt{\frac{a}{x}-1}+C=-\frac{2}{a}\sqrt{\frac{a-x}{x}}+C$

4. $\dfrac{1}{x^2(x^4+1)^{3/4}}$

SOLUTION

Let $I=\int \frac{dx}{x^2(x^4+1)^{3/4}}=\int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$ Put $1+\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt \Rightarrow \frac{1}{x^5}dx=-\frac{1}{4}dt \therefore I=-\frac{1}{4}\int \frac{dt}{t^{3/4}}=-\frac{1}{4}\int t^{-3/4}dt$

$$=-\frac{1}{4}\frac{t^{1/4}}{1/4}+C=-(t)^{1/4}+C=-\left(1+\frac{1}{x^4}\right)^{1/4}+C$$

5. $\dfrac{1}{x^{1/2}+x^{1/3}}$ [Hint : $\frac{1}{x^{1/2}+x^{1/3}}=\frac{1}{x^{1/3}\left(1+x^{1/6}\right)}$, Put $x=t^6$]

SOLUTION

: Let $I=\int \frac{dx}{x^{1/2}+x^{1/3}}$ L.C.M of 2 and 3 is 6 So Put $x=t^6 \Rightarrow dx=6t^5dt \therefore I=\int \frac{6t^5dt}{t^3+t^2}=6\int \frac{t^3}{t+1}dt$ Since the degree of numerator is greater than the denominator, $\therefore$ the fraction is improper. First, we mark it proper by dividing t^3 by $t+1$.

$=6\int\left[t^2-t+1-\frac{1}{t+1}\right]dt=6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\log|t+1|\right]+C=2\sqrt{x}-3x^{1/3}+6x^{1/6}-6\log\left|x^{1/6}+1\right|+C$

6. $\frac{5x}{(x+1)(x^2+9)}$

SOLUTION

Let $I=\int \frac{5x\,dx}{(x+1)(x^2+9)}$ Let $\frac{5x}{(x+1)(x^2+9)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+9} \Rightarrow 5x=A\left(x^2+9\right)+(Bx+C)(x+1)$ ....(i) Putting $x=-1$ in (i), we get $5(-1)=A(1+9)\Rightarrow A=-\frac{1}{2}$ Comparing coefficients of x^2 in (i) , we get $0=A+B \Rightarrow B=\frac{1}{2}$ Putting $x=0$ in (i), we get$0=9A+C\Rightarrow C=-9A=-9\left(-\frac{1}{2}\right)=\frac{9}{2} \therefore I=\int \frac{-\frac{1}{2}}{x+1}dx+\int \frac{\frac{1}{2}x+\frac{9}{2}}{x^2+9}dx=-\frac{1}{2}\log(x+1)+\frac{1}{4}\int \frac{2x}{x^2+9}dx+\frac{9}{2}\int \frac{dx}{x^2+3^2}+C$

$=-\frac{1}{2}\log(x+1)+\frac{1}{4}\log\left(x^2+9\right)+\frac{9}{2}\times\frac{1}{3}\tan^{-1}\frac{x}{3}+C=-\frac{1}{2}\log(x+1)+\frac{1}{4}\log\left(x^2+9\right)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$

7. $\frac{\sin x}{\sin(x-a)}$

SOLUTION

Let $I=\int \frac{\sin x}{\sin(x-a)}dx=\int \frac{\sin((x-a)+a)}{\sin(x-a)}dx=\int \frac{\sin(x-a)\cos a+cos(x-a)\sin a}{\sin(x-a)}dx=\cos a\int(1)\,dx+\sin a\int \cot(x-a)\,dx$
$=x\cos a+\sin a\log|\sin(x-a)|+C$

8. $\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

SOLUTION

Let $I=\int \frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx=\int\left(\frac{e^{\log x^5}-e^{\log x^4}}{e^{\log x^3}-e^{\log x^2}}\right)dx$

$=\int \frac{x^5-x^4}{x^3-x^2}dx=\int \frac{x^4(x-1)}{x^2(x-1)}dx=\int x^2dx=\frac{x^3}{3}+C$

9. $\frac{\cos x}{\sqrt{4-\sin^2x}}$

SOLUTION

Let $I=\int \frac{\cos x}{\sqrt{4-\sin^2x}}dx$ Let $\sin x=t \Rightarrow \cos x dx=dt \therefore I=\int \frac{dt}{\sqrt{4-t^2}}=\sin^{-1}\left(\frac{t}{2}\right)+C=\sin^{-1}\left(\frac{\sin x}{2}\right)+C$

10. $\frac{\sin^8x-\cos^8x}{1-2\sin^2x\cos^2x}$

SOLUTION

Let $I=\int \frac{\sin^8x-\cos^8x}{1-2\sin^2x\cos^2x}dx$ We have, $\left(\sin^8x-\cos^8x\right)=\left(\sin^4x+\cos^4x\right)\left(\sin^4x-\cos^4x\right)=\left[\left(\sin^2x+\cos^2x\right)^2-2\sin^2x\cos^2x\right]\left(\sin^2x+\cos^2\right.$

$=\left(1-2\sin^2x\cos^2x\right)(1)(-\cos 2x) \therefore I=\int \frac{\left(1-2\sin^2x\cos^2x\right)(-\cos 2x)}{1-2\sin^2x\cos^2x}dx=-\int \cos 2x dx=-\frac{1}{2}\sin 2x+C$

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11. $\dfrac{1}{\cos(x+a)\cos(x+b)}$

SOLUTION

Let $I=\displaystyle\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}dx$
$=\dfrac{1}{\sin(a-b)}[\log|\sec(x+a)|-\log|\sec(x+b)|]+C=\dfrac{1}{\sin(a-b)}\log\left|\dfrac{\sec(x+a)}{\sec(x+b)}\right|+C=\dfrac{1}{\sin(a-b)}\log\left|\dfrac{\cos(x+b)}{\cos(x+a)}\right|+C$

12. . $\dfrac{x^3}{\sqrt{1-x^8}}$

SOLUTION

Let $I=\displaystyle\int\frac{x^3}{\sqrt{1-x^8}}dx=\frac{1}{4}\int\frac{4x^3}{\sqrt{1-(x^4)^2}}dx$ Put $x^4=t\Rightarrow 4x^3dx=dt\therefore I=\dfrac{1}{4}\displaystyle\int\frac{dt}{\sqrt{1-t^2}}=\frac{1}{4}\sin^{-1}(t)+C=\frac{1}{4}\sin^{-1}\left(x^4\right)+C$

13. . $\dfrac{e^x}{(1+e^x)(2+e^x)}$

SOLUTION

Let $I=\displaystyle\int\frac{e^x}{(1+e^x)(2+e^x)}dx$ Put $e^x=t\Rightarrow e^xdx=dt\therefore I=\displaystyle\int\frac{dt}{(1+t)(2+t)}$ Now, we write $\dfrac{1}{(1+t)(2+t)}=\dfrac{A}{1+t}+\dfrac{B}{2+t}\Rightarrow$ $1=A(2+t)+B(1+t)$ ....(i) Putting $t=-1$ in (i), we get $1=A(2-1)\Rightarrow A=1$ Putting $t=-2$ in (i), we get $1=B(1-2)$ $\Rightarrow B=-1\therefore I=\displaystyle\int\frac{1}{(1+t)(2+t)}dt=\int\left(\frac{1}{1+t}-\frac{1}{2+t}\right)dt=\log(1+t)-\log(2+t)+C=\log(1+e^x)-\log(2+e^x)+C=$ $\log\left(\dfrac{1+e^x}{2+e^x}\right)+C$

14. . $\dfrac{1}{(x^2+1)(x^2+4)}$

SOLUTION

: Let $I=\displaystyle\int\frac{1}{(x^2+1)(x^2+4)}dx$ Now, consider $\dfrac{1}{(x^2+1)(x^2+4)}$ Put$x^2=t$ We write, $\dfrac{1}{(t+1)(t+4)}=\dfrac{A}{(t+1)}+\dfrac{B}{t+4}\Rightarrow 1=$ $A(t+4)+B(t+1)$ ....(i) Putting $t=-1$ in (i), we get $1=A(-1+4)\Rightarrow A=\dfrac{1}{3}$ Putting $t=-4$ in (i), we get $1=B(-4+1)\Rightarrow$ $B=-\dfrac{1}{3}\therefore\dfrac{1}{(t+1)(t+4)}=\dfrac{1}{3(t+1)}-\dfrac{1}{3(t+4)}=\dfrac{1}{3(x^2+1)}-\dfrac{1}{3(x^2+4)}$ Now, $I=\displaystyle\int\left[\frac{1}{3(x^2+1)}-\frac{1}{3(x^2+4)}\right]dx=\left(\frac{1}{3}\tan^{-1}x\right)-$ $\left(\dfrac{1}{3}\times\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)\right)+C=\dfrac{1}{3}\tan^{-1}x-\dfrac{1}{6}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

15. $\cos^3 xe^{\log\sin x}$

SOLUTION

Let $I=\displaystyle\int\cos^3 xe^{\log\sin x}dx=\int\cos^3 x\sin xdx$ Put $\cos x=t\Rightarrow-\sin xdx=dt\therefore I=-\displaystyle\int t^3dt=-\frac{t^4}{4}+C=-\frac{1}{4}\cos^4 x+C$

16. . $e^{3\log x}\left(x^4+1\right)^{-1}$

SOLUTION

.: Let $I=\displaystyle\int e^{3\log x}\left(x^4+1\right)^{-1}dx=\int e^{\log x^3}\left(x^4+1\right)^{-1}dx=\int x^3\left(x^4+1\right)^{-1}dx=\int\frac{x^3}{x^4+1}dx$ Let $x^4=t\Rightarrow 4x^3dx=dt\therefore I=$ $\dfrac{1}{4}\displaystyle\int\frac{dt}{t+1}=\frac{1}{4}\log(t+1)+C=\frac{1}{4}log\left(x^4+1\right)+C$

17. $f'(ax+b)[f(ax+b)]^n$

SOLUTION

Let $I=\int f'(ax+b)[f(ax+b)]^n dx$ Let $f(ax+b)=t \Rightarrow af'(ax+b)\,dx=dt \therefore I=\frac{1}{a}\int t^n dt=\frac{1}{a}\cdot\frac{t^{n+1}}{n+1}+C=\frac{1}{(n+1)a}[f(ax+b)]^{n+1}+$ $C \Rightarrow$

18. $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$

SOLUTION

: Let $I=\int\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx=\int\sqrt{\frac{\sin x}{\sqrt{\sin^4 x\sin(x+\alpha)}}}dx=\int\frac{1}{\sin^2 x}\sqrt{\frac{\sin x}{\sin(x+\alpha)}}dx$ Let $\frac{\sin(x+\alpha)}{\sin x}=t\Rightarrow\frac{\sin x\cos(x+\alpha)-\cos x\sin}{\sin^2 x}$ $dt \Rightarrow \frac{\sin[x-(x+\alpha)]}{\sin^2 x}dx=dt\Rightarrow-\frac{\sin\alpha}{\sin^2 x}dx=dt \therefore I=\int-\frac{1}{\sin\alpha}\cdot\frac{1}{\sqrt{t}}dt=-\frac{1}{\sin\alpha}\int t^{-1/2}dt=-\frac{1}{\sin\alpha}\left(\frac{t^{1/2}}{1/2}\right)+C=\frac{-2}{\sin\alpha}\sqrt{t}+$ $C=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}+C$

19. $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x\in[0,1]$

SOLUTION

: Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx=\int\frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx=\frac{2}{\pi}\int\left[2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right]dx=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}dx-\int(1)\,dx$ $\ldots$(i) Let $I_1=\int\sin^{-1}\sqrt{x}dx$ Put $\sqrt{x}=t\Rightarrow x=t^2\Rightarrow dx=2t\,dt \therefore I_1=2\int\sin^{-1}t\cdot t\,dt=2\left[\sin^{-1}t\int t\,dt-\int\left(\frac{d}{dt}(\sin^{-1}t)\int t\,dt\right)dt\right]$ $=2\left[\sin^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{\sqrt{1-t^2}}\frac{t^2}{2}dt\right]+C_1=t^2\sin^{-1}t+\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt+C_1=t^2\sin^{-1}t+\int\sqrt{1-t^2}dt-\int\frac{1}{\sqrt{1-t^2}}dt+C_1=t^2\sin^{-1}t+$ $\left(\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right)-\sin^{-1}t+C_1=x\sin^{-1}\sqrt{x}+\frac{\sqrt{x}\sqrt{1-x}}{2}-\frac{1}{2}\sin^{-1}\sqrt{x}+C_1$ $=\left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x}+\frac{\sqrt{x}\sqrt{1-x}}{2}+C_1 \therefore$ Form (i) we have $I=\frac{4}{\pi}\left(\frac{2x-1}{2}\right)\sin^{-1}\sqrt{x}+\frac{2\sqrt{x}\sqrt{1-x}}{\pi}-x+C\left[C=\frac{4}{\pi}C_1\right]=\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+$ $\frac{2\sqrt{x}\sqrt{1-x}}{\pi}-x+C$

20. . $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

SOLUTION

: Let $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ Put $\sqrt{x}=\cos t\Rightarrow x=\cos^2 t\Rightarrow dx=2\cos t(-\sin t)\,dt \therefore I=\int\sqrt{\frac{1-\cos t}{1+\cos t}}(-2\sin t\cos t)\,dt=-4\int\sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}}\sin\frac{t}{2}$ $=-4\int\sin^2\frac{t}{2}\cos t\,dt=-4\int\frac{1-\cos t}{2}\cos t\,dt=-2\int(\cos t-\cos^2 t)\,dt=-2\int\left(\cos t-\frac{1+\cos 2t}{2}\right)dt=-\int(2\cos t-1-\cos 2t)\,dt=$ $-\left[2\sin t-t-\frac{\sin 2t}{2}\right]+C=-[2\sin t-t-\sin t\cos t]+C=-\left[2\sqrt{1-x}-\cos^{-1}\sqrt{x}-\sqrt{1-x}\sqrt{x}\right]+C=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+$ $\sqrt{x}\sqrt{1-x}+C$

21. . $\frac{2+\sin 2x}{1+\cos 2x}e^x$

SOLUTION

Let $I=\int \frac{e^x(2+\sin 2x)}{1+\cos 2x}dx=\int e^x\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)dx=\int e^x\left[\sec^2 x+\tan x\right]dx=\int e^x\left(\tan x+\sec^2 x\right)dx=\int e^x\left[\tan x+\frac{d}{dx}(\tan x)\right]dx=$ $e^x\tan x+C$

22. . $\frac{x^2+x+1}{(x+1)^2(x+2)}$

SOLUTION

: Let$I=\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$ We write, $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)}\Rightarrow x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$ ....(i) Putting $x=-1$ in (i), we get $1-1+1=B(-1+2)\Rightarrow B=1$ Putting $x=-2$ in (i), we get $4-2+1=C(-2+1)^2$ $\Rightarrow C=3$ Putting $x=0$ in (i), we get $1=2A+2B+C\Rightarrow 1=2A+2+3\Rightarrow A=-2\Rightarrow I=\int\left[\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right]dx$

$=-2\log(x+1)+\frac{(x+1)^{-1}}{-1}+3\log(x+2)+C=-2\log(x+1)-\frac{1}{x+1}+3\log(x+2)+C$

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

SOLUTION

Let $I=\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$ Let $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta\therefore I=\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx=-\int\tan^{-1}\left(\tan\frac{\theta}{2}\right)(\sin\theta)d\theta=-\int\frac{\theta}{2}\sin\theta d\theta=-$

$=-\frac{1}{2}\left[\theta(-\cos\theta)-\int 1(-\cos\theta)d\theta\right]=\frac{1}{2}\theta\cos\theta-\frac{1}{2}\int\cos\theta d\theta=\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta+C=\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sqrt{1-\cos^2\theta}+C=$ $\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+C$

24. . $\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

SOLUTION

Let $I=\int\sqrt{x^2+1}\frac{\left[\log(x^2+1)-2\log x\right]}{x^4}dx=\int\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]\frac{dx}{x^3}=\int\sqrt{1+\frac{1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]\frac{dx}{x^3}$ Put $\frac{1}{x^2}=t\Rightarrow$

$-2x^{-3}dx=dt\Rightarrow-\frac{2}{x^3}dx=dt\therefore I=\frac{-1}{2}\int\sqrt{1+t}\log(1+t)dt\therefore I=\frac{-1}{2}\left[\log(1+t)\cdot\frac{(1+t)^{3/2}}{3/2}-\int\frac{1}{1+t}\frac{(1+t)^{3/2}}{3/2}dt\right]=-\frac{1}{2}\left[\frac{2}{3}(1+t)^{3/2}1\right.$

$=-\frac{1}{2}\left[\frac{2}{3}(1+t)^{3/2}\log(1+t)-\frac{2}{3}\frac{(1+t)^{3/2}}{3\,3/2}\right]+C=-\frac{1}{2}\left[\frac{2}{3}(1+t)^{3/2}\log(1+t)-\frac{4}{9}(1+t)^{3/2}\right]+C=-\frac{1}{3}(1+t)^{3/2}\log(1+t)+$ $\frac{2}{9}(1+t)^{3/2}+C=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\log\left(1+\frac{1}{x^2}\right)+\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\left(\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right)\right]+C$

Evaluate the definite integrals in Exercises 25 to 33. :

25. . $\int_{\pi/2}^{\pi}e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

SOLUTION Let $I=\int_{\pi/2}^{\pi}e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx=\int_{\pi/2}^{\pi}e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx=\int_{\pi/2}^{\pi}e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)dx=-\int_{\pi/2}^{\pi}e^x\left(\cot\frac{x}{2}-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right.$

$=-\left[e^x\cot\frac{x}{2}\right]_{\pi/2}^{\pi}=-\left[e^{\pi}\cot\frac{\pi}{2}-e^{\pi/2}\cot\frac{\pi}{4}\right]=-\left[0-e^{\pi/2}\cdot 1\right]=e^{\pi/2}$

26. $\int\limits_{0}^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

SOLUTION

Let $I = \int\limits_{0}^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$ Dividing numerator and denominator by $\cos^4 x$,we get $I = \int\limits_{0}^{\pi/4} \frac{\tan x \sec^2 x dx}{1+\tan^4 x}$ Put $\tan^4 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$ When $x = 0, t = 0$ and when $x = \frac{\pi}{4}, t = 1 \therefore I = \frac{1}{2}\int\limits_{0}^{1} \frac{dt}{1+t^2} = \left[\frac{1}{2}\tan^{-1} t\right]_0^1 = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$

27. $\int\limits_{0}^{\pi/2} \frac{\cos^2 x dx}{\cos^2 x + 4\sin^2 x}$

SOLUTION

Let $I = \int\limits_{0}^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx = \int\limits_{0}^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4(1-\cos^2 x)} dx = \int\limits_{0}^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x} dx = -\frac{1}{3}\int\limits_{0}^{\pi/2} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx = -\frac{1}{3}\int\limits_{0}^{\pi/2} \left(1 - \frac{4}{4-3\cos^2 x}\right) dx =$ $= -\frac{1}{3}\left(\frac{\pi}{2}\right) + \frac{4}{3}\int\limits_{0}^{\pi/2} \frac{\sec^2 x}{4\sec^2 x - 3} dx = -\frac{\pi}{6} + \frac{4}{3}\int\limits_{0}^{\pi/2} \frac{\sec^2 x}{4(1+\tan^2 x) - 3} dx$ Put $\tan x = t \Rightarrow \sec^2 x dx = dt$ When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty \therefore I = -\frac{\pi}{6} + \frac{4}{3}\int\limits_{0}^{\infty} \frac{dt}{4(1+t^2) - 3} = \frac{\pi}{6} + \frac{4}{3}\int\limits_{0}^{\infty} \frac{dt}{4t^2 + 1} = -\frac{\pi}{6} + \frac{4}{3}\cdot\frac{1}{4}\int\limits_{0}^{\infty} \frac{dt}{t^2 + \frac{1}{4}} = -\frac{\pi}{6} + \frac{1}{3}\cdot\frac{1}{1/2}\left[\tan^{-1}\frac{t}{1/2}\right]_0^{\infty} = -\frac{\pi}{6} + \frac{2}{3}\left[\tan^{-1} 2t\right]_0^{\infty} = -\frac{\pi}{6} + \frac{2}{3}\left[\tan^{-1}\infty - \tan^{-1} 0\right] = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2} - 0\right] = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$

28. $\int\limits_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

SOLUTION

Let $I = \int\limits_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \int\limits_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(1-\sin 2x)}} dx \int\limits_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$ Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$ When $x = \frac{\pi}{6}, t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2}$ and When $x = \frac{\pi}{3}, t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} \therefore I = \int\limits_{\frac{1}{2}-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}-\frac{1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{\frac{1}{2}-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$ $= \sin^{-1}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 2\sin^{-1}\frac{1}{2}\left(\sqrt{3} - 1\right)$

29. $\int\limits_{0}^{1} \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

SOLUTION

Let $I = \int\limits_{0}^{1} \frac{dx}{\sqrt{1+x} - \sqrt{x}} = \int\limits_{0}^{1} \frac{\sqrt{1+x} + \sqrt{x}}{1+x-x} dx = \int\limits_{0}^{1} \left[\sqrt{1+x} + \sqrt{x}\right] dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1 + \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}\left(2^{3/2} - 1\right) + \frac{2}{3} = \frac{2}{3}\cdot 2^{3/2} - \frac{2}{3} + \frac{2}{3} = \frac{2}{3}\cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3}$

30. $\int\limits_{0}^{\pi/4} \frac{\text{sinx} + cosx}{9 + 16\sin 2x} dx$

SOLUTION

Let $I=\int_0^{\pi/4}\frac{\sin x+cosx}{9+16\sin 2x}dx$ Put $\sin x-\cos x=t\Rightarrow(\cos x+\sin x)\,dx=dt$ And $1-2\sin x\cos x=t^2\Rightarrow 1-\sin 2x=t^2$ When $x=\frac{\pi}{4},t=sin\frac{\pi}{4}-\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$ When $x=0,t=\sin 0-\cos 0=-1\therefore\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx=\int_{-1}^{0}\frac{dt}{9+16\left(1-t^2\right)}=\int_{-1}^{0}\frac{dt}{25-16t^2}$ $=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}=\frac{1}{16}\cdot\frac{1}{2\times\frac{5}{4}}\left[\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}=\frac{1}{40}[\log 1-(\log 1-\log 9)]=\frac{1}{40}\log 9$

31. $\int_0^{\pi/2}\sin 2x\tan^{-1}(\sin x)\,dx$

SOLUTION

: Let $I=\int_0^{\pi/2}\sin 2x\tan^{-1}(\sin x)\,dx\Rightarrow I=\int_0^{\pi/2}2\sin x\cos x\tan^{-1}(\sin x)\,dx$ Put $\sin x=t\Rightarrow\cos x dx=dt$ When $x=0,t=0$ and when $x=\frac{\pi}{2},t=1\therefore I=2\int_0^1 t\tan^{-1}t\,dt=2\left[\tan^{-1}t\int t dt-\int\left(\frac{d}{dt}\left(\tan^{-1}t\right)\cdot\int t dt\right)dt\right]_0^1=2\left[\tan^{-1}(t)\frac{t^2}{2}-\int\frac{1}{1+t^2}\cdot\frac{t^2}{2}dt\right]_0^1=$ $2\left[\frac{t^2}{2}\tan^{-1}(t)-\frac{1}{2}\int\frac{1+t^2-1}{1+t^2}\cdot dt\right]_0^1]=\left[t^2\tan^{-1}(t)-\int\left(1-\frac{1}{1+t^2}\right)dt\right]_0^1=\left[t^2\tan^{-1}(t)-t+\tan^{-1}t\right]_0^1=\tan^{-1}(1)-1+\tan^{-1}1=$ $\frac{\pi}{4}-1+\frac{\pi}{4}=\frac{\pi}{2}-1$

32. . $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$

SOLUTION

: Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx\ldots$(i) Also, $I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)(-\tan x)}{(-\sec x)+(-\tan x)}dx=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}dx\ldots$.(ii)

Adding (i) and (ii), we get $2I=\int_0^{\pi}\frac{(x+\pi-x)\tan x}{\sec x+tan x}=\pi\int_0^{\pi}\frac{\tan x}{\sec x+tan x}dx=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx=\pi\int_0^{\pi}\frac{(1+\sin x)-1}{1+\sin x}dx=$ $\pi\int_0^{\pi}\left(1-\frac{1}{1+\sin x}\right)dx=\pi\int_0^{\pi}(1)\,dx-\pi\int_0^{\pi}\frac{1-\sin x}{1-\sin^2 x}dx=\pi[x]_0^{\pi}-\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx=\pi(\pi-0)-\pi\int_0^{\pi}\left(\sec^2 x-\sec x\tan x\right)dx=$ $\pi^2-\pi[\tan x-\sec x]_0^{\pi}=\pi^2-\pi[(\tan\pi-\tan 0)-(\sec\pi-\sec 0)]=\pi^2-\pi(0-0)+\pi(-1-1)=\pi^2-2\pi=\pi(\pi-2)\therefore I=\frac{\pi}{2}(\pi-2)$

33. . $\int_1^4 [|x-1|+|x-2|+|x-3|]dx$

SOLUTION

: Let $I=\int_1^4 (|x-1|+|x-2|+|x-3|)dx$ $|x-1|=x-1$, when $x\geq 1$ $|x-2|=x-2$, when $x\geq 2$ $|x-2|=-(x-2)$, when $x\leq 2$ $|x-3|=-(x-3)$, when $x\leq 3$ $|x-3|=(x-3)$, when $x\geq 3$ $\Rightarrow I=\int_1^4 (x-1)\,dx-\int_1^2 (x-2)\,dx+\int_1^4 (x-2)\,dx-\int_1^3 (x-3)\,dx+\int_1^4 (x-3)\,dx=\left[\frac{x^2}{2}-x\right]_1^4-\left[\frac{x^2}{2}-2x\right]_1^2+\left[\frac{x^2}{2}-2x\right]_2^4-\left[\frac{x^2}{2}-3x\right]_1^3+\left[\frac{x^2}{2}-3x\right]_3^4=\left[\left(\frac{16}{2}-\frac{1}{2}\right)-(4-1)\right]-\left[\left(\frac{4}{2}-\frac{1}{2}\right)-(4-2)\right]+\left[\left(\frac{16}{2}-\frac{4}{2}\right)-(8-4)\right]-\left[\left(\frac{9}{2}-\frac{1}{2}\right)-(9-3)\right]+\left[\left(\frac{16}{2}-\frac{9}{2}\right)-(12-9)\right]=\left[\frac{15}{2}-\frac{3}{2}+\frac{12}{2}-\frac{8}{2}+\frac{7}{2}\right]+[-3+2-4+6-3]=\left[\frac{23}{2}\right]+[-2]=\frac{19}{2}$

Prove the following (Exercises 34 to 39) :

34. . $\int_1^3 \frac{dx}{x^2(x+1)}=\frac{2}{3}+\log\frac{2}{3}$

SOLUTION

: Let $\frac{1}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}\Rightarrow 1=Ax(x+1)+B(x+1)+Cx^2$ ....(i) Putting $x=0$ in (i), we get $1=B(0+1)\Rightarrow B=1$ Putting $x=-1$ in (i), we get $1=C(-1)^2\Rightarrow C=1$ Comparing coefficients of x^2 on both sided of (i), we get $0=A+C\Rightarrow A=-C$ $\Rightarrow A=-1\therefore \frac{1}{x^2(x+1)}=-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\therefore I=\int_1^3\frac{dx}{x^2(x+1)}=\int_1^3\left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx=\left[-\log|x|+\frac{x^{-1}}{-1}+\log|x+1|\right]_1^3=\left[-\frac{1}{x}+\log\left|\frac{x+1}{x}\right|\right]_1^3=\left(-\frac{1}{3}+1\right)+\log\frac{4}{3}-\log 2=\frac{2}{3}+\log\left(\frac{4}{3}\times\frac{1}{2}\right)=\frac{2}{3}+\log\frac{2}{3}$

35. $\int_0^1 xe^x dx=1$

SOLUTION

: Let $I=\int_0^1 xe^x dx$ Integrating by parts, $I=\left[x\int e^x dx-\int\left(\frac{d}{dx}(x)\int e^x dx\right)dx\right]_0^1=\left[xe^x-\int e^x dx\right]_0^1=[xe^x-e^x]_0^1=[(e-0)-(e-1)]=1$

36. $\int_{-1}^1 x^{17}\cos^4 x dx=0$ SOLUTION

Let $I=\int_{-1}^1 x^{17}\cos^4 xdx$ Let $f(x)=x^{17}\cos^4 x\Rightarrow f(-x)=(-x)^{17}\cos^4(-x)=-x^{17}\cos^4 x=-f(x)\therefore f(x)$ is an odd function , hence $I=0$

37. . $\int_0^{\pi/2}\sin^3 xdx=\frac{2}{3}$

SOLUTION

: Let $\int\limits_0^{\pi/2} \sin^3 x dx = \frac{1}{4}\int\limits_0^{\pi/2} (3\sin x - \sin 3x)\, dx$

$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi/2} = \frac{1}{4}\left[-3\cos\frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right] - \frac{1}{4}\left[-3\cos 0 + \frac{\cos 0}{3}\right] = \frac{1}{4}\left[0+0+3-\frac{1}{3}\right] = \frac{1}{4}\left(\frac{8}{3}\right) = \frac{2}{3}$

38. $\int\limits_0^{\pi/4} 2\tan^3 x dx = 1 - \log 2$

SOLUTION

: $\int\limits_0^{\pi/4} 2\tan^3 x dx = \int\limits_0^{\pi/4} 2\tan x \cdot \tan^2 dx = \int\limits_0^{\pi/4} 2\tan x\left(\sec^2 x - 1\right) dx = 2\int\limits_0^{\pi/4} (\tan x)\sec^2 x dx - 2\int\limits_0^{\pi/4} \tan x\, dx = 2\left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} - 2\left[-\log|\cos x|\right]_0^{\pi/4}$

$= \left(\tan^2\frac{\pi}{4} - \tan^2 0\right) + 2\left(\log\cos\frac{\pi}{4} - \log\cos 0\right) = (1-0) + 2\left(\log\frac{1}{\sqrt{2}} - \log 1\right) = 1 + 2\left(\log 1 - \log\sqrt{2} - \log 1\right) = 1 + 2\times\left(-\frac{1}{2}\log 2\right) =$
$1 - \log 2$

39. $\int\limits_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$

SOLUTION

: $\int\limits_0^1 \sin^{-1} x dx = \int\limits_0^1 \sin^{-1} x \cdot 1 dx$ Integrating by parts $= \left[\sin^{-1} x \int dx - \left(\int \frac{d}{dx}(\sin^{-1} x)\cdot \int (1) dx\right) dx\right]_0^1 = \left[\sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \times x dx\right]_0^1$

$= \left[x\sin^{-1} x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} dx\right]_0^1 = \left[x\sin^{-1} x + \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right]\right]_0^1 = \left[(\sin^{-1} 1) + \frac{1}{2}\times\frac{2}{1}(0-1)\right] = \frac{\pi}{2} - 1$

40. . Evaluate $\int\limits_0^1 e^{2-3x} dx$ as a limit of a sum.

SOLUTION

: Let $I = \int\limits_a^b e^{2-3x} dx$ Here $a = 0, b = 1$ $f(x) = e^{2-3x}$ $h = \frac{1-0}{n} = \frac{1}{n} \Rightarrow nh = 1$ $f(a) = f(0) = e^2$ $f(a+h) = f(0+h) = e^{2-3h}$
$f(a+2h) = f(0+2h) = e^{2-6h}$ ....

$f(a+(n-1)h) = f(a+(n-1h)) = e^{3(n-1)h}$ $\therefore \int\limits_0^1 f(x)\, dx = \lim\limits_{h\to 0} h\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right] = \lim\limits_{h\to 0}\left[e^2 + e^{2-3h} + e^{2-6h} + ..\right.$

$= \lim\limits_{h\to 0}\left[e^2\left(1 + e^{-3h} + e^{-6h} + ... + e^{-3(n-1)h}\right)\right] = \lim\limits_{h\to 0}\left[e^2\left(\frac{1\left(e^{-3hn}-1\right)}{e^{-3h}-1}\right)\right] = \lim\limits_{h\to 0}\left[e^2\left(\frac{h\left(e^{-3}-1\right)}{e^{-3h}-1}\right)\right] = \lim\limits_{h\to 0}\left[e^2\left(\frac{-3h}{e^{-3h}-1}\times\frac{e^{-3}-1}{-3}\right)\right.$

$\frac{e^2\times\left(e^{-3}-1\right)}{-3} = \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$

Choose the correct answer in Exercises 41 to 44. :

41. $\int \frac{dx}{e^x + e^{-x}}$ is equal to

(a) $\tan^{-1}\left(e^x\right) + C$
(b) $\tan^{-1}\left(e^{-x}\right) + C$
(c) $\log\left(e^x - e^{-x}\right) + C$
(d) $\log\left(e^x + e^{-x}\right) + C$

SOLUTION

: Let $I=\int \frac{dx}{e^x+e^{-x}}=\int \frac{dx}{e^x+\frac{1}{e^x}} \Rightarrow \int \frac{e^x dx}{(e^x)^2+1}$ Put $e^x=t \Rightarrow e^x dx=dt \therefore I=\int \frac{dt}{1+t^2}=\tan^{-1}(t)+C=\tan^{-1}(e^x)+C$

42. $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is equal to

(a) $\frac{-1}{\sin x+\cos x}+C$
(b) $\log|\sin x+\cos x|+C$
(c) $\log|\sin x-\cos x|+C$
(d) $\frac{1}{(\sin x+\cos x)^2}+C$

SOLUTION

(B) : Let $I=\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx=\int \frac{\cos^2 x-\sin^2 x}{(\cos x-\sin x)^2}dx=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx=\log(\cos x+\sin x)+C$

43. If $f(a+b-x)=f(x)$, then $\int\limits_a^b xf(x)\,dx$ is equal to

(a) $\frac{a+b}{2}\int\limits_a^b f(b-x)\,dx$
(b) $\frac{a+b}{2}\int\limits_a^b f(b+x)\,dx$
(c) $\frac{b-a}{2}\int\limits_a^b f(x)\,dx$
(d) $\frac{a+b}{2}\int\limits_a^b f(x)\,dx$

SOLUTION

Let $I=\int\limits_a^b xf(x)\,dx$ Let $a+b-x=z \Rightarrow -dx=dz$ When $x=a, z=b$ and when $x=b, z=a \therefore I=-\int\limits_b^a (a+b-z)f(z)\,dz=\int\limits_a^b (a+b)f(z)\,dz-\int\limits_a^b zf(z)\,dz=(a+b)\int\limits_a^b f(x)\,dx-\int\limits_a^b xf(x)\,dx=(a+b)\int\limits_a^b f(x)\,dx-I \Rightarrow 2I=(a+b)\int\limits_a^b f(x)\,dx$ Hence, $I=\left(\frac{a+b}{2}\right)\int\limits_a^b f(x)dx$

44. The value of $\int\limits_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

(a) 1
(b) 0
(c) -1
(d) $\frac{\pi}{4}$

SOLUTION

Let $I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx=\int_0^1 \tan^{-1}\left[\frac{x+x-1}{1-x(x-1)}\right]dx=\int_0^1 \left[\tan^{-1}x+\tan^{-1}(x-1)\right]dx \Rightarrow I=\int_0^1 \tan^{-1}xdx+\int_0^1 \tan^{-1}(x-1)\,dx=$

$I_1+I_2\ldots$.(i) Where, $I_1=\int_0^1 1\cdot\tan^{-1}xdx$ Integrating by parts $=\left[\tan^{-1}x\int(1)\,dx-\int\left(\frac{d}{dx}(\tan^{-1}x)\cdot\int(1)dx\right)dx\right]_0^1=\left[x\tan^{-1}x-\int\frac{1}{1+x^2}\right.$

$=\left[x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx\right]_0^1=\left[x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)\right]_0^1$

$=\left[\left(\tan^{-1}1-0\right)-\frac{1}{2}(\log 2-\log 1)\right]=\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]$ .....(ii) And $I_2=\int_2^1 1\cdot\tan^{-1}(x-1)\,dx$ Again integrating by parts $=\left[\tan^{-1}(x-1)\int(\right.$

$=\left[\tan^{-1}(x-1)\cdot x-\int\left(\frac{1}{1+(x-1)^2}\times x\right)dx\right]_0^1=\left[x\tan^{-1}(x-1)-\frac{1}{2}\int\frac{2(x-1+1)}{1+(x-1)^2}dx\right]_0^1=\left[x\tan^{-1}(x-1)-\frac{1}{2}\int\frac{2(x-1)\,dx}{1+(x-1)^2}-\frac{1}{2}\int\right.$

$=\left[x\tan^{-1}(x-1)\right]_0^1-\frac{1}{2}\left[\log\left(1+(x-1)^2\right)\right]_0^1-\left[\tan^{-1}(x-1)\right]_0^1=[0-0]-\frac{1}{2}[0-\log 2]-\left[0+\frac{\pi}{4}\right]=\frac{1}{2}\log 2-\frac{\pi}{4}\ldots$(iii) From (i),

(ii) and (iii) we get $I=\left[\frac{\pi}{4}-\frac{1}{2}\log 2+\frac{1}{2}\log 2-\frac{\pi}{4}\right]=0$