NCERT - Exercise -7.5

Integrate the rational functions in Exercise 1 to 21.:

1. $\dfrac{x}{(x+1)(x+2)}$

SOLUTION

.: Let $I=\int \dfrac{x\,dx}{(x+1)(x+2)}$ Let $\dfrac{x}{(x+1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x+2} \Rightarrow x=A(x+2)+B(x+1)\ldots$(i) Putting $x=-1$ in (i), we get $-1=A(-1+2)\Rightarrow A=-1$ Putting $x=-2$ in (i), we get $-2=B(-2+1)\Rightarrow B=2 \therefore \dfrac{x}{(x+1)(x+2)}=\dfrac{-1}{x+1}+\dfrac{2}{x+2}$

$\Rightarrow I=\int \dfrac{x}{(x+1)(x+2)}dx=\int\left[\dfrac{-1}{x+1}+\dfrac{2}{x+2}\right]dx$

$=\int \dfrac{-1}{(x+1)}dx+\int \dfrac{2}{x+2}dx=-\log|x+1|+2\log|x+2|+C=-\log|x+1|+\log(x+2)^2+C=\log\left[\dfrac{(x+2)^2}{|x+1|}\right]+C$

2. $\dfrac{1}{x^2-9}$

SOLUTION

Let $I=\int \dfrac{dx}{x^2-9}$ Let $\dfrac{1}{x^2-9}=\dfrac{1}{(x-3)(x+3)}=\dfrac{A}{x-3}+\dfrac{B}{x+3}\Rightarrow 1=A(x+3)+(x-3)\ \ldots$(i) Putting $x=3$ in (i), we get $1=A(3+3)\Rightarrow A=\dfrac{1}{6}$ Putting $x=-3$ in (i), we get $1=B(-3-3)\Rightarrow B=-\dfrac{1}{6}\therefore I=\int\dfrac{dx}{x^2-9}=\dfrac{1}{6}\int\left[\dfrac{1}{x-3}-\dfrac{1}{x+3}\right]dx$

$=\dfrac{1}{6}[\log|x-3|-\log|x+3|]+C=\dfrac{1}{6}\log\left|\dfrac{x-3}{x+3}\right|+C$

3. $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

SOLUTION

Let $I=\int\dfrac{(3x-1)\,dx}{(x-1)(x-2)(x-3)}$ We write, $\dfrac{3x-1}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}\Rightarrow 3x-1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\ \ldots$(i) Putting $x=1$ in (i), we get $3-1=A(1-2)(1-3)\Rightarrow 2=A(-1)(-2)\Rightarrow A=1$ Putting $x=2$ in (i), we get $6-1=B(2-1)(2-3)\Rightarrow 5=B(1)(-1)\Rightarrow B=-5$ Putting $x=3$ in (i), we get $9-1=C(3-1)(3-2)\Rightarrow 8=C(2)(1)\Rightarrow C=4\therefore\dfrac{3x-1}{(x-1)(x-2)(x-3)}=\dfrac{1}{x-1}-\dfrac{5}{x-2}+\dfrac{4}{x-3}\Rightarrow I=\int\dfrac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\dfrac{1}{x-1}dx-5\int\dfrac{1}{x-2}dx+4\int\dfrac{dx}{x-3}$
$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$

4. $\dfrac{x}{(x-1)(x-2)(x-3)}$

SOLUTION

.: Let $I=\int\dfrac{x\,dx}{(x-1)(x-2)(x-3)}$ We write, $\dfrac{x}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}\Rightarrow x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ ..(i) Putting $x=1$ in (i), we get $1=A(1-2)(1-3)\Rightarrow A=\dfrac{1}{2}$ Putting $x=2$ in (i), we get $2=B(2-1)(2-3)\Rightarrow B=-2$ Putting $x=3$ in (i), we get $3=C(3-1)(3-2)\Rightarrow C=\dfrac{3}{2}\therefore\dfrac{x}{(x-1)(x-2)(x-3)}=\dfrac{1}{2(x-1)}-\dfrac{2}{x-2}+\dfrac{3}{2(x-3)}\therefore$
$\int\dfrac{x}{(x-1)(x-2)(x-3)}dx=\dfrac{1}{2}\int\dfrac{dx}{x-1}-2\int\dfrac{dx}{x-2}+\dfrac{3}{2}\int\dfrac{dx}{x-3}=\dfrac{1}{2}\log|x-1|-2\log|x-2|+\dfrac{3}{2}\log|x-3|+C$

5. $\dfrac{2x}{x^2+3x+2}$

SOLUTION

: Let $I=\int \frac{2x}{x^2+3x+2}dx$ We write, $\frac{2x}{x^2+3x+2}=\frac{2x}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2} \Rightarrow 2x=A(x+2)+B(x+1)$ ..(i) Putting $x=-1$ in (i), we get $2(-1)=A(-1+2) \Rightarrow A=-2$ Putting $x=-2$ in (i), we get $2(-2)=B(-2+1) \Rightarrow B=4 \therefore \frac{2x}{x^2+3x+2}=\frac{-2}{x+1}+\frac{4}{x+2} \Rightarrow I=\int \frac{2x}{x^2+3x+2}dx=-2\int \frac{dx}{x+1}+4\int \frac{dx}{x+2}=-2\log|x+1|+4\log|x+2|+C$

6. $\dfrac{1-x^2}{x(1-2x)}$

SOLUTION

: Since $\frac{1-x^2}{x(1-2x)}=\frac{1-x^2}{x-2x^2}$ is an improper fraction, therefore, we convert it into a proper fraction by long division method . we get $\frac{x^2-1}{2x^2-x}=\frac{1}{2}+\frac{\frac{x}{2}-1}{2x^2-x} \Rightarrow I=\int \frac{(-1+x^2)}{-x+2x^2}dx=\frac{1}{2}\int dx+\frac{1}{2}+\int \frac{x-2}{2x^2-x}dx$ Now, $\frac{x-2}{2x^2-x}=\frac{x-2}{x(2x-1)}=\frac{A}{x}+\frac{B}{2x-1} \Rightarrow x-2=A(2x-1)+Bx \ldots$(i) Putting $x=0$ in (i), we get $-2=A(-1) \Rightarrow A=2$ Putting $x=\frac{1}{2}$ in (i), we get $\frac{1}{2}-2=B\left(\frac{1}{2}\right) \Rightarrow 1-4=B \Rightarrow B=-3 \therefore \frac{x-2}{2x^2-x}=\frac{2}{x}-\frac{3}{2x-1}=\frac{2}{x}+\frac{3}{1-2x} \Rightarrow I=\int \frac{1-x^2}{x(1-2x)}dx=\frac{1}{2}\int (1)\,dx+\frac{1}{2}\int \left(\frac{2}{x}+\frac{3}{1-2x}\right)dx=\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C$

7. $\dfrac{x}{(x^2+1)(x-1)}$

SOLUTION

Let $I=\int \frac{xdx}{(x^2+1)(x-1)}$ We write, $\frac{x}{(x^2+1)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1} \Rightarrow x=A\left(x^2+1\right)+(Bx+C)(x-1) \ldots$.(i)

Putting $x=1$ in (i) ,we get $1=A(1+1) \Rightarrow A=\frac{1}{2}$ Comparing coefficients of x^2 in (i), we get $0=A+B \Rightarrow B=-\frac{1}{2}$ Comparing the constant terms, we get $0=A-C \Rightarrow C=\frac{1}{2} \therefore I=\int \frac{x}{(x^2+1)(x-1)}dx=\int \left[\frac{1/2}{x-1}+\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\right]dx=\frac{1}{2}\int \frac{dx}{x-1}-\frac{1}{2}\int \frac{x}{x^2+1}dx+\frac{1}{2}\int \frac{dx}{x^2+1}$

$=\frac{1}{2}\log(x-1)-\frac{1}{4}\int \frac{2x}{x^2+1}+\frac{1}{2}\tan^{-1}x+C=\frac{1}{2}\log(x-1)-\frac{1}{4}\log\left(x^2+1\right)+\frac{1}{2}\tan^{-1}x+C$

8. $\dfrac{x}{(x-1)^2(x+2)}$

SOLUTION

: Let $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} \Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \ldots.(\text{i})$

Comparing coefficients of x^2 on both sides of (i), we get $0 = A + C$ Putting $x = -2$ in (i) ,we get $-2 = C(-2-1)^2 \Rightarrow C = \dfrac{-2}{9} \Rightarrow$ $A = -C = \dfrac{2}{9}$ Putting $x = 1$ in (i) ,we get $1 = B(1+2) \Rightarrow B = \dfrac{1}{3} \therefore \displaystyle\int \frac{x}{(x-1)^2(x+2)}dx = \int \frac{2}{9(x-1)}dx + \int \frac{1}{3(x-1)^2}dx - \int \frac{2}{9(x+2)}dx$
$= \dfrac{2}{9}\log|x-1| + \dfrac{1}{3} \times \dfrac{(x-1)^{-1}}{-1} - \dfrac{2}{9}\log|x+2| + C = \dfrac{2}{9}\log\left|\dfrac{x-1}{x+2}\right| - \dfrac{1}{3(x-1)} + C$

9. $\dfrac{3x+5}{x^3-x^2-x+1}$

SOLUTION

: Let $I = \displaystyle\int \frac{3x+5}{x^3-x^2-x+1}dx$ Also, $\dfrac{3x+5}{x^2(x-1)-1(x-1)} = \dfrac{3x+5}{(x^2-1)(x-1)} = \dfrac{3x+5}{(x+1)(x-1)^2}$ We write, $\dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{A}{x-1} +$ $\dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} \Rightarrow 3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \ldots.(\text{i})$ Putting $x = 0$ in (i) ,we get $5 = -A + B + C \ldots.(\text{ii})$
Putting $x = 1$ in (i) ,we get $3+5 = B \times 2 \Rightarrow B = \dfrac{8}{2} = 4$ Putting $x = -1$ in (i) ,we get $-3+5 = C(-1-1)^2 \Rightarrow C = \dfrac{2}{4} = \dfrac{1}{2}$ Now, Putting values of B and C in (ii), we get $5 = -A + 4 + \dfrac{1}{2} \Rightarrow A = \dfrac{9}{2} - 5 = -\dfrac{1}{2} \therefore I = \displaystyle\int \frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int \frac{dx}{(x-1)} + 4\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int$
$= \dfrac{1}{2}\log|x-1| + 4\left(-\dfrac{1}{x-1}\right) + \dfrac{1}{2}\log|x+1| + C = \dfrac{1}{2}\log\left|\dfrac{x+1}{x-1}\right| - \dfrac{4}{x-1} + C$
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10. $\dfrac{2x-3}{(x^2-1)(2x+3)}$

SOLUTION

: Let $I = \displaystyle\int \frac{2x-3}{(x^2-1)(2x+3)}dx$ We write, $\dfrac{2x-3}{(x^2-1)(2x+3)} = \dfrac{2x-3}{(x-1)(x+1)(2x+3)} \Rightarrow \dfrac{2x-3}{(x-1)(x+1)(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} +$ $\dfrac{C}{2x+3} \Rightarrow 2x-3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) \ldots.(\text{i})$ Putting $x = 1$ in (i) ,we get $2(1) - 3 = A(1+1)(2+3) \Rightarrow -$
$A(2)(5) \Rightarrow A = -\dfrac{1}{10}$ Putting $x = -1$ in (i) ,we get $-2-3 = B(-1-1)(-2+3) \Rightarrow -5 = B(-2)(1) \Rightarrow B = \dfrac{5}{2}$ Putting $x = -\dfrac{3}{2}$ in (i) ,we get $-3-3 = C\left(-\dfrac{3}{2}-1\right)\left(-\dfrac{3}{2}+1\right) \Rightarrow -6 = C\left(-\dfrac{5}{2}\right)\left(-\dfrac{1}{2}\right) \Rightarrow C = -6 \times \dfrac{4}{5} = \dfrac{-24}{5} \therefore \dfrac{2x-3}{(x^2-1)(2x+3)} = -\dfrac{1}{10(x-1)} +$ $\dfrac{5}{2(x+1)} - \dfrac{24}{5(2x+3)} \Rightarrow I = \displaystyle\int \frac{2x-3}{(x^2-1)(2x+3)}dx = -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3} = -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| -$ $\dfrac{24}{5 \times 2}\log|2x+3| + C = \dfrac{5}{2}\log|x+1| - \dfrac{1}{10}\log|x-1| - \dfrac{12}{5}\log|2x+3| + C$

11. $\dfrac{5x}{(x+1)(x^2-4)}$

SOLUTION

Let $I = \displaystyle\int \frac{5x\,dx}{(x+1)(x^2-4)}dx$ We write , $\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5x}{(x+1)(x+2)(x-2)}$ $\dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$ $\Rightarrow 5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$ ..(i) Putting $x = -1$ in (i), we get $-5 = A(-1+2)(-1-2) \Rightarrow$ $A = \dfrac{5}{3}$ Putting $x = 2$ in (i), we get $10 = C(2+1)(2+2) \Rightarrow C = \dfrac{5}{6} \therefore \dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5}{3(x+1)} - \dfrac{5}{2(x+2)} + \dfrac{5}{6(x-2)} \therefore I =$ $\displaystyle\int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int \frac{dx}{x+1} - \frac{5}{2}\int \frac{dx}{x+2} + \frac{5}{6}\int \frac{dx}{x-2} = \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$

12. $\frac{x^3+x+1}{x^2-1}$

SOLUTION

: Let $I=\int\frac{x^3+x+1}{x^2-1}dx$ Since $\frac{x^3+x+1}{x^2-1}$ is an improper fraction, therefore, we convert it into proper fraction by long division method. $\therefore\ \frac{x^2+x+1}{x^2-1}=Q+\frac{R}{D}\ \therefore\ \frac{x^2+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$ ....(i) Now, we write $\frac{2x+1}{x^2-1}=\frac{2x+1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}\Rightarrow$ $2x+1=A(x-1)+B(x+1)$ .....(ii) Putting $x=-1$ in (ii), we get $-2+1=A(-1-1)\Rightarrow A=\frac{-1}{-2}=\frac{1}{2}$ Putting $x=-1$ in (ii), we get $2+1=B(1+1)\Rightarrow B=\frac{3}{2}\therefore\frac{2x+1}{x^2-1}=\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$ ....(iii) From (i) and (iii), $\frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$
$\therefore I=\int\frac{x^3+x+1}{x^2-1}dx=\int xdx+\frac{1}{2}\int\frac{dx}{x+1}+\frac{3}{2}\int\frac{dx}{x-1}=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$

13. $\frac{2}{(1-x)(1+x^2)}$

SOLUTION

: Let $I=\int\frac{2}{(1-x)(1+x^2)}dx$ We write, $\frac{2}{(1-x)(1+x^2)}=\frac{A}{1-x}+\frac{Bx+C}{1+x^2}\Rightarrow 2=A(1+x^2)+(Bx+C)(1-x)$ ...(i) Putting $x=1$ in (i), we get $2=A(1+1)\Rightarrow A=1$ Comparing coefficients of x^2 on both sides of (i), we get $0=A-B\Rightarrow A=B$ $\Rightarrow B=1$ ...(ii) Comparing coefficients of constant terms on both sides of (i), we get $2=A+C\Rightarrow C=2-1=1$ ...(iii) $\therefore\ \frac{2}{(1-x)(1+x^2)}=\frac{1}{1-x}+\frac{x+1}{1+x^2}\therefore I=\int\frac{2}{(1-x)(1+x^2)}dx=\int\left(\frac{1}{1-x}+\frac{x+1}{1+x^2}\right)dx=\int\frac{1}{1-x}dx+\int\frac{xdx}{1+x^2}+\int\frac{1}{1+x^2}dx$
$=-\log|1-x|+\frac{1}{2}\log\left|1+x^2\right|+\tan^{-1}x+C$

14. $\frac{3x-1}{(x+2)^2}$

SOLUTION

: Let $I=\int\frac{3x-1}{(x+2)^2}dx$ We write, $\frac{3x-1}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}\Rightarrow 3x-1=A(x+2)+B$ ...(i) Comparing coefficients of x in (i), we get $A=3$ Comparing the constant terms in (i), we get $2A+B=-1\Rightarrow B=-7\ \therefore\ \frac{3x-1}{(x+2)^2}=\frac{3}{x+2}+\frac{-7}{(x+2)^2}\Rightarrow I=$ $\int\frac{3x-1}{(x+2)^2}dx=3\int\frac{dx}{x+2}-7\int\frac{dx}{(x+2)^2}=3\log|x+2|-7\frac{(x+2)^{-1}}{-1}+C=3\log|x+2|+\frac{7}{x+2}+C$

15. $\frac{1}{x^4-1}$

SOLUTION

.: Let $I=\int\frac{1}{x^4-1}dx$ Let $\frac{1}{x^4-1}=\frac{1}{(x+1)(x-1)(x^2+1)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}\Rightarrow 1=A(x-1)\left(x^2+1\right)+B(x+1)\left(x^2+1\right)+$ $(Cx+D)(x+1)(x-1)$ ...(i) Putting $x=-1$ in (i), we get $1=A(-1-1)(1+1)\Rightarrow 1=A(-4)\Rightarrow A=-\frac{1}{4}$ Putting $x=1$ in (i), we get $1=B(1+1)(1+1)\Rightarrow 1=B(2)(2)\Rightarrow B=\frac{1}{4}$ Comparing coefficients of x^3 and constant in (i) on both sides , we get $0=A+B+C$ and $-A+B-D=1\Rightarrow 0=\frac{-1}{4}+\frac{1}{4}+C\Rightarrow C=0$ and $1=\frac{1}{4}+\frac{1}{4}-D\Rightarrow D=-\frac{1}{2}\therefore\ \frac{1}{x^4-1}=\frac{1}{4(x+1)}+$ $\frac{1}{4(x-1)}+\frac{-1}{2(x^2+1)}\Rightarrow I=\int\frac{1}{x^4-1}dx=-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{4}\int\frac{1}{x-1}-\frac{1}{2}\int\frac{1}{(x^2+1)}dx=-\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$
$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

16. $\frac{1}{x(x^n+1)}$

[Hint : multiply numerator and denominator by x^{n-1} and Put $x^n = t$]

SOLUTION

: Let $I = \int \frac{dx}{x(x^n+1)}$ Now $\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x \cdot x^{n-1}(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$ Put $x^n = t \Rightarrow nx^{n-1}dx = dt \therefore \int \frac{dx}{x(x^n+1)} = \frac{1}{n} \cdot \int \frac{nx^{n-1}}{x^n(x^n+1)}dx + \frac{1}{n}\int \frac{dt}{t(t+1)}$ ....(i) We write, $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \Rightarrow 1 = A(t+1) + Bt$ ...(ii) Putting $t = 0$ in (ii), we get $1 = A(0+1) \Rightarrow A = 1$ Putting $t = -1$ in (ii), we get $1 = B(-1) \Rightarrow B = -1 \therefore I = \int \frac{dx}{x(x^n+1)} = \frac{1}{n}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \frac{1}{n}[\log|t| - \log|t+1|] + C = \frac{1}{n}\log\left|\frac{t}{t+1}\right| + C = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$

17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint : Put $\sin x = t$]

SOLUTION

: Let $I = \int \frac{\cos x dx}{(1-\sin x)(2-\sin x)}$ Put $\sin x = t \Rightarrow \cos x dx = dt \therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{1}{(1-t)(2-t)}dt$ We write $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t} \Rightarrow 1 = A(2-t) + B(1-t)$ ...(i) Putting $t = 1$ in (i), we get $1 = A(2-1) \Rightarrow A = 1$ Putting $t = 2$ in (i), we get $B = -1 \therefore I = \int \frac{\cos x dx}{(1-\sin x)(2-\sin x)} = \int \frac{1}{1-t}dt - \int \frac{dt}{2-t} = -\log|1-t| + \log|2-t| + C = \log\left|\frac{2-t}{1-t}\right| + C = \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

SOLUTION

Let $I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx$ Put $x^2 = t \Rightarrow \frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12} = 1 + \frac{-4t-10}{t^2+7t+12} = 1 - \left[\frac{(4t+10)}{(t+3)(t+4)}\right]$ We write , $\frac{4t+10}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4} \Rightarrow 4t + 10 = A(t+4) + B(t+3)$ ....(i) Putting $t = -3$ in (i) , we get $4(-3) + 10 = A(-3+4) \Rightarrow A = -2$ Putting $t = -4$ in (i) , we get $4(-4) + 10 = B(-4+3) \Rightarrow B = 6 \therefore \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left[\frac{-2}{t+3} + \frac{6}{t+4}\right] = 1 + \left[\frac{2}{x^2+3} - \frac{6}{x^2+4}\right] \therefore I = \int \left[1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}\right] dx = x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right) + C = x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$

19. $\frac{2x}{(x^2+1)(x^2+3)}$

SOLUTION

Let $I = \int \frac{2x}{(x^2+1)(x^2+3)}dx$ Put $x^2 = y \Rightarrow 2xdx = dy \therefore I = \int \frac{dy}{(y+1)(y+3)}$ We write, $\frac{1}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3}$ $1 = A(y+3) + B(y+1)$ ...(i) Putting $y = -1$ in (i) ,we get $1 = 2A \Rightarrow A = \frac{1}{2}$ Putting $y = -3$ in (i) ,we get $1 = -2B \Rightarrow B = -\frac{1}{2} \Rightarrow \frac{1}{(y+1)(y+3)} = \frac{1}{2(y+1)} - \frac{1}{2(y+3)} \therefore I = \int \left[\frac{1}{2(y+1)} - \frac{1}{2(y+3)}\right] dy = \frac{1}{2}\int \frac{dy}{y+1} - \frac{1}{2}\int \frac{dy}{y+3} = \frac{1}{2}\log|y+1| - \frac{1}{2}\log|y+3| + C = \frac{1}{2}\log\left|\frac{y+1}{y+3}\right| + C = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$

20. $\frac{1}{x(x^4-1)}$

SOLUTION

$\therefore$ Let $I=\int \frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int \frac{4x^3dx}{x^4\left(x^4-1\right)}$ Put $x^4=t\Rightarrow 4x^3dx=dt$ $\therefore I=\frac{1}{4}\int \frac{dt}{t\left(t-1\right)}$ We write, $\frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{t-1}\Rightarrow 1=A\left(t-1\right)+Bt$ ...(i) Putting $t=0$ in (i) ,we get $1=A\left(-1\right)\Rightarrow A=-1$ Putting $t=1$ in (i) ,we get $1=B\left(1\right)\Rightarrow B=1$ $\therefore$ $\frac{1}{t\left(t-1\right)}=\frac{-1}{t}+\frac{1}{t-1}\Rightarrow I=\frac{1}{4}\int\left(\frac{-1}{t}+\frac{1}{t-1}\right)dt=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$

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21. $\frac{1}{\left(e^x-1\right)}$ [Hint : Put $e^x=t$]

SOLUTION

$\therefore$ Let $I=\int \frac{1}{e^x-1}dx$ Put $e^x=t\Rightarrow e^xdx=dt\Rightarrow dx=\frac{dt}{t}$ $\therefore I=\int \frac{dt}{t\left(t-1\right)}$ We write, $\frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{t-1}\Rightarrow 1=A\left(t-1\right)+Bt$ ..(i) Putting $t=1$ in (i) ,we get $B=1$ Putting $t=0$ in (i) ,we get $1=A\left(0-1\right)\Rightarrow A=-1$ $\therefore$ $\frac{1}{t\left(t-1\right)}=\frac{-1}{t}+\frac{1}{t-1}\Rightarrow I=\int\left(\frac{-1}{t}+\frac{1}{t-1}\right)dt=-\log|t|+\log|t-1|+C=-\log|e^x|+\log|e^x-1|+C=\log\left|\frac{e^x-1}{e^x}\right|+C$

Choose the correct answer in each of the Exercises 22 and 23.:

22. $\int \frac{x\,dx}{\left(x-1\right)\left(x-2\right)}$ equals

(a) $\log\left|\frac{\left(x-1\right)^2}{x-2}\right|+C$

(b) $\log\left|\frac{\left(x-2\right)^2}{x-1}\right|+C$

(c) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

(d) $\log\left|\left(x-1\right)\left(x-2\right)\right|+C$

SOLUTION

: $I=\int \frac{x}{\left(x-1\right)\left(x-2\right)}dx$ We write $\frac{x}{\left(x-1\right)\left(x-2\right)}=\frac{A}{x-1}+\frac{B}{x-2}\Rightarrow x=A\left(x-2\right)+B\left(x-1\right)$ Comparing like terms, we get $A+B=1$ and $-2A-B=0$ Solving, we get $A=-1,B=2$ $\therefore I=\int\left(\frac{-1}{x-1}+\frac{2}{x-2}\right)dx=-\log|x-1|+2\log|x-2|+C=\log\left|\frac{\left(x-2\right)^2}{\left(x-1\right)}\right|+C$

23. $\int \frac{dx}{x\left(x^2+1\right)}$ equals

(a) $\log|x|-\frac{1}{2}\log\left(x^2+1\right)+C$

(b) $\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

(c) $-\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

(d) $\frac{1}{2}\log|x|+\log\left(x^2+1\right)+C$

SOLUTION

: (A) Let $\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1} \Rightarrow 1=A\left(x^2+1\right)+(Bx+C)(x)$ ….(i) Putting $x=0$ in (i), we get $1=A(0+1) \Rightarrow A=1$ Comparing coefficients of x^2 in (i) on both sides , we get $0=A+B \Rightarrow B=-1$ Comparing coefficients of x in (i) on both sides , we get$C=0 \therefore \int \frac{1}{x\left(x^2+1\right)}dx=\int\left[\frac{1}{x}+\frac{-x}{x^2+1}\right]dx=\log x-\frac{1}{2}\log\left(x^2+1\right)+C$