NCERT Exercise- 7.11

By using the properties of definite integrals, evaluate the integrals in Exercise 1 to 19. :

1. $\int\limits_0^{\pi/2} \cos^2 x dx$

SOLUTION Let $I = \int\limits_0^{\pi/2} \cos^2 x dx \ldots$(i) and $I = \int\limits_0^{\pi/2} \cos^2\left(\frac{\pi}{2}-x\right) dx \ldots$(ii)

Adding (i) and (ii), we have $2I = \int\limits_0^{\pi/2} \cos^2 x dx + \int\limits_0^{\pi/2} \sin^2 x dx = \int\limits_0^{\pi/2} \left(sin^2 x + \cos^2 x\right) dx = \int\limits_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

2. $\int\limits_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

SOLUTION : Let$I = \int\limits_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \ldots$(i)

$\Rightarrow I = \int\limits_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx = \int\limits_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \ldots$(ii) Adding (i) and (ii), we have $2I = \int\limits_0^{\pi/2} \left[\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{}{\sqrt{c}}\right.$

$= \int\limits_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx = \int\limits_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

3. $\int^{\pi/2} \frac{\sin^{3/2} x dx}{\sin^{3/2} x + \cos^{3/2} x}$

SOLUTION Let $I = \int\limits_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx \ldots$(i) $\Rightarrow I = \int\limits_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right) + \cos^{3/2}\left(\frac{\pi}{2}-x\right)} dx = \int\limits_0^{\pi/2} \frac{\cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$

$\ldots$(ii)

Adding (i) and (ii), we have $2I = \int\limits_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx = \int\limits_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \therefore I = \frac{\pi}{4}$

4. $\int\limits_0^{\pi/2} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$

SOLUTION Let$I = \int\limits_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx \ldots$(i) Also, $I = \int\limits_0^{\pi/2} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right) + \cos^5\left(\frac{\pi}{2}-x\right)} dx = \int\limits_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \ldots$(ii)

Adding (i) and (ii), we have $2I = \int\limits_0^{\pi/2} \frac{\cos^2 x}{\cos^5 x + \sin^5 x} dx + \int\limits_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx = \int\limits_0^{\pi/2} \frac{\cos^5 x + \sin^5 x}{\cos^5 x + \sin^5 x} dx = \int\limits_1^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2} \therefore I = \frac{\pi}{4}$

5. $\int_{-5}^{5} |x+2|dx$

SOLUTION Let $I = \int_{-5}^{5} |x+2|dx$ $|x+2| = \begin{cases} -(x+2), & \text{if } x < -2 \\ x+2, & \text{if } x \geq -2 \end{cases}$ $\therefore I = -\int_{-5}^{-2} (x+2)\,dx + \int_{-2}^{5} (x+2)\,dx = -\left[\frac{(x+2)^2}{2}\right]_{-5}^{-2} +$

$\left[\frac{(x+2)^2}{2}\right]_{-2}^{5} = -\left[\left(\frac{(-2+2)^2}{2} - \frac{(-5+2)^2}{2}\right)\right] + \left[\frac{(5+2)^2}{2} - \frac{(-2+2)^2}{2}\right] = -\frac{1}{2}[0-9] + \frac{1}{2}[49-0] = \frac{9}{2} + \frac{49}{2} = \frac{58}{2} = 29$

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6. $\int_{2}^{8} |x-5|dx$

SOLUTION Let$I=\int_{2}^{8}|x-5|dx=\int_{2}^{5}|x-5|\,dx+\int_{2}^{8}|x-5|dx\; |x-5|=\begin{cases} -(x-5), & \text{if} x<5 \\ (x-5), & \text{if} x\geq 5 \end{cases}\;\therefore I=-\int_{2}^{5}(x-5)\,dx+\int_{5}^{8}(x-5)\,dx$

$=-\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8=-\frac{1}{2}(25-4)+5(5-2)+\frac{1}{2}(64-25)-5(8-5)=\frac{-21}{2}+15+\frac{39}{2}-15=\frac{18}{2}=9$

7. $\int_{0}^{1} x(1-x)^n dx$

SOLUTION Let $I=\int_{0}^{1}x(1-x)^n dx\Rightarrow I=\int_{0}^{1}(1-x)\,[1-(1-x)]^n dx$

$=\int_{0}^{1}(1-x)\,x^n dx=\int_{0}^{1}\left(x^n-x^{n+1}\right)dx=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1=\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-(0-0)=\frac{1}{(n+1)\,(n+2)}$

8. $\int_{0}^{\pi/4} \log(1+\tan x)\,dx$

SOLUTION

Let$I=\int_{0}^{\pi/4}\log(1+\tan x)\,dx\ldots$(i) Also, $I=\int_{0}^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$\Rightarrow I=\int_{0}^{\pi/4}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx=\int_{0}^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx=\int_{0}^{\pi/4}\log 2dx-\int_{0}^{\pi/4}\log(1+\tan x)\,dx=\log 2\int_{0}^{\pi/4}1\,dx-I=2I=\log 2\,[x]_0^{\pi/4}=$
$(\log 2)\left(\frac{\pi}{4}-0\right)\Rightarrow I=\frac{\pi}{8}\log 2$

9. $\int_{0}^{2} x\sqrt{2-x}dx$

SOLUTION

Let$I=\int_{0}^{2}x\sqrt{2-x}dx$ Put $2-x=t\Rightarrow -dx=dt$ When $x=0,t=2$ and when $x=2,t=0\therefore I=-\int_{2}^{0}(2-t)\sqrt{t}dt=\int_{0}^{2}\left(2t^{1/2}-t^{3/2}\right)dt=$
$\left[\frac{2t^{3/2}}{3/2}-\frac{t^{5/2}}{5/2}\right]_0^2$

$=\left[\frac{4}{3}t^{3/2}-\frac{2}{5}t^{5/2}\right]_0^2=\frac{4}{3}(2)^{3/2}-\frac{2}{5}(2)^{5/2}=\frac{4}{3}\times 2\sqrt{2}-\frac{2}{5}\times 4\sqrt{2}=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}=\frac{16\sqrt{2}}{15}$

10. $\int_{0}^{\pi/2}(2\log sin x-log\,sin\,2x)dx$

SOLUTION

Let$I=\int_{0}^{\pi/2}(2\log sin x-log\,sin\,2x)dx=\int_{0}^{\pi/2}[2\log sin x-log\,(2sin x\cos x)]dx=\int_{0}^{\pi/2}[2\log sin x-log 2-\log\sin x-\log\cos x]dx=\int_{0}^{\pi/2}\log\sin xd$
$\int_{0}^{\pi/2}\log\cos\left(\frac{\pi}{2}-x\right)dx$

$$= \int_0^{\pi/2} \log\sin x dx - (\log 2)[x]_0^{\pi/2} - \int_0^{\pi/2} \log \sin x \, dx = -(\log 2)\left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log(2)^{-1} = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

11. . $\int_{-\pi/2}^{\pi/2} \sin^2 x dx$

SOLUTION

Let $I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx$ Let $f(x) = \sin^2 x \Rightarrow f(x) = f(-x) \therefore f(x)$ is an even function. $\therefore I = 2\int_0^{\pi/2} \sin^2 x dx$

$$= 2\int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \left[\frac{\pi}{2} - \frac{\sin \pi}{2}\right] \Rightarrow I = \frac{\pi}{2}$$

12. . $\int_0^{\pi} \frac{x dx}{1+\sin x}$

SOLUTION

Let $I = \int_0^{\pi} \frac{x dx}{1+\sin x} \ldots\ldots (i) \Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \ldots (ii)$ Adding (i) and (ii), we get $2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x} dx =$
$\pi \int_0^{\pi} \frac{1}{1+\sin x} dx = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx = \pi \int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx = \pi [\tan x - \sec x]_0^{\pi} = \pi [(\tan \pi - \sec \pi) - (\tan 0 - \sec 0)]$
$\pi[(0) - (-1) - (0-1)] = 2\pi$ Hence, $I = \pi$

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13. $\int_{-\pi/2}^{\pi/2} \sin^7 x dx$

SOLUTION

Let $f(x) = \sin^7 x \Rightarrow f(-x) = [\sin(-x)]^7 = (-\sin(x))^7 = -f(x) \Rightarrow f(x)$ is an odd function of x. $\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$

14. $\int_0^{2\pi} \cos^5 x dx$

SOLUTION

Let $I = \int_0^{2\pi} \cos^5 x dx$,Let $f(x) = \cos^5 x$ Now, we have $f(2\pi - x) = (\cos(2\pi - x))^5 = (\cos x)^5 = \cos^5 x = f(x)$ $\left[\int_0^{2a} f(x) dx = \begin{cases} 2\int_0^a f(x) dx, \\ 0, \quad \text{if } f \end{cases}\right.$

$\Rightarrow I = 2\int_0^{\pi} \cos^5 x dx$ Again , we have $f(\pi - x) = (\cos(\pi - x))^5 = -\cos^5 x = -f(x) \Rightarrow 2\int_0^{\pi} \cos^5 x dx = 0$ Hence, $\int_0^{2\pi} \cos^5 x dx = 2\int_0^{\pi} \cos^5 x dx = 2$

15. $\int_{0.}^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

SOLUTION

Let$I = \int_{0.}^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ .....(i) Then $I = \int_{0.}^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$

$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \ldots$(ii) Adding (i) and (ii), we get $2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \sin x \cos x}\right) dx = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x} dx =$
$\int_0^{\pi/2} 0 dx = 0 \Rightarrow I = 0$

16. $\int_0^{\pi} \log(1 + \cos x) dx$

SOLUTION

Let $I = \int_0^{\pi} \log(1 + \cos x) dx \ldots$(i) $\Rightarrow I = \int_0^{\pi} \log[1 + \cos(\pi - x)] dx = \int_0^{\pi} \log(1 - \cos x) dx \ldots$.(ii)

Adding (i) and (ii), we get $2I = \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)] dx = \int_0^{\pi} \log\left(1 - \cos^2 x\right) dx = \int_0^{\pi} \log \sin^2 x dx = 2\int_0^{\pi} \log \sin x dx \Rightarrow$

$I = \int_0^{\pi} \log \sin x dx = 2\int_0^{\pi} \log \sin x dx = 2I_1$

Where $I_1 = \int_0^{\pi/2} \log \sin x dx \ldots$.(iii) Then, $I_1 = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow I_1=\int_0^{\pi/2}\log\cos x dx$ (iv) Adding (iii) and (iv), we get $2I_1=\int_0^{\pi/2}\log\sin x dx+\int_0^{\pi/2}\log\cos x dx=\int_0^{\pi/2}(\log\sin x+\log\cos x)\,dx=\int_0^{\pi/2}\log(\sin x\cos x)\,dx=\int_0^{\pi/2}\log\left(\frac{2\sin x\cos x}{2}\right)dx=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx=\int_0^{\pi/2}\log\sin 2x dx-\int_0^{\pi/2}\log 2 dx=\int_0^{\pi/2}\log\sin 2x dx-(\log 2)\,[x]_0^{\pi/2}=\int_0^{\pi/2}\log\sin 2x dx-(\log 2)\left(\frac{\pi}{2}-0\right)=\int_0^{\pi/2}\log\sin 2x dx-\frac{\pi}{2}\log 2=I_2-\frac{\pi}{2}-\log 2$ .....(V) Where $I_2=\int_0^{\pi/2}\log\sin 2x dx$ Put $2x=t$ $\Rightarrow 2dx=dt$ When $x=0,t=0$ and when $x=\frac{\pi}{2},t=\pi\therefore I_2=\frac{1}{2}\int_0^{\pi}\log\sin t dt=\frac{1}{2}\cdot 2\int_0^{\pi/2}\log\sin t\,dt$

$=\int_0^{\pi/2}\log\,sin x dx=I_1\therefore$ From (V), we get $2I_1=I_1-\frac{\pi}{2}\log 2\Rightarrow I_1=-\frac{\pi}{2}\log 2\therefore I=2\times\left(-\frac{\pi}{2}\log 2\right)=-\pi\log 2$

17. . $\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

SOLUTION

: Let $I=\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ ....(i) $\Rightarrow I=\int_0^a\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}dx=\int_0^a\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$ ....(ii) Adding (i) and (ii), we get $2I=\int_0^a\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx=\int_0^a(1)\,dx=[x]_0^a=a-0=a\therefore I=\frac{a}{2}$

18. $\int_0^4|x-1|\,dx$

SOLUTION

Let $I=\int_0^4|x-1|\,dx$ $|x-1|=\begin{cases}-(x-1), & if\,x<1\\ x-1, & if\,x\geq 1\end{cases}$ $\therefore I=-\int_0^1(x-1)\,dx+\int_1^4(x-1)\,dx=-\left[\frac{(x-1)^2}{2}\right]_0^1+\left[\frac{(x-1)^2}{2}\right]_1^4=-\frac{1}{2}[0-1]+\frac{1}{2}[9-0]=\frac{1}{2}+\frac{9}{2}=5$

19. Show that $\int_0^a f(x)\,g(x)\,dx=2\int_0^a f(x)\,dx$,if f and g are defined as $f(x)=f(a-x)$ and $g(x)+g(a-x)=4$

SOLUTION

Let $I=\int_0^a f(x)\,g(x)\,dx=\int_0^a f(a-x)\,[4-g(a-x)]\,dx=4\int_0^a f(a-x)\,dx-\int_0^a f(a-x)\,g(a-x)\,dx$ Let $a-x=t\Rightarrow -dx=dt$ When $x=0,t=a$and when $x=a,t=0\Rightarrow I=-4\int_a^0 f(t)\,dt+\int_a^0 f(t)\,g(t)\,dt=4\int_0^a f(t)\,dt-\int_0^a f(t)\,g(t)\,dt=4\int_0^a f(x)\,dx-\int_0^a f(x)\,g(x)\,dx=4\int_0^a f(x)\,dx-I\Rightarrow 2I=4\int_0^a f(x)\,dx$ Hence, $I=2\int_0^a f(x)\,dx$

Choose the correct answer in Exercises 20 to 21.:

20. The value of $\int_{-\pi/2}^{\pi/2}\left(x^3+x\cos x+\tan^5 x+1\right)dx$ is

(a) 0

(b) 2

(c) π

(d) 1

SOLUTION

$I=\int_{-\pi/2}^{\pi/2}\left(x^3+x\cos x+\tan^5 x+1\right)dx \Rightarrow I=\int_{-\pi/2}^{\pi/2}\left(x^3+x\cos x+\tan^5 x+1\right)dx+\int_{-\pi/2}^{\pi/2}1\,dx$ $I=I_1+[x]_{-\pi/2}^{\pi/2}=I_1+\frac{\pi}{2}+\frac{\pi}{2}$ Where $I_1=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x\right)dx \Rightarrow I=I_1+\pi$ Now, for I_1 , let $f(x)=x^3+x\cos x+\tan^5 x \therefore f(-x)=(-x)^3+(-x)\cos(-x)+\tan^5(-x)$ $=\left(-x^3-x\cos x-\tan^5 x\right)=-f(x) \therefore f(x)$ is an odd function Thus, $I_1=0$, Hence, $I=\pi$

21. The value of $\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

(a) 2

(b) 3/4

(c) 0

(d) -2

SOLUTION

$I=\int_0^{\pi/2}\log\left[\frac{4+3\sin x}{4+3\cos x}\right]dx$ Also, $I=\int_0^{\pi/2}\log\left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right]dx$

$\Rightarrow I=\int_0^{\pi/2}\log\left[\frac{4+3\cos x}{4+3\sin x}\right]dx \Rightarrow I=-\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \Rightarrow I=-I \Rightarrow 2I=0 \Rightarrow I=0$