NCERT Exercise – 7.10

Evaluate the definite integrals in Exercise 1 to 8. Using substitution.:

1. $\int\limits_0^1 \frac{x}{x^2+1}dx$

SOLUTION

Let $I=\int\limits_0^1 \frac{x}{x^2+1}dx$ Put $x^2+1=t \Rightarrow 2xdx=dt$ When $x=0, t=1$ and when $x=1, t=2 \therefore I=\frac{1}{2}\int\limits_1^2 \frac{dt}{t}=\left[\frac{1}{2}\log t\right]_1^2=\frac{1}{2}[\log 2-\log 1]=\frac{1}{2}\log 2$

2. $\int\limits_0^{\pi/2} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

SOLUTION : Let $I=\int\limits_0^{\pi/2} \sqrt{\sin\phi}\cos^5\phi\, d\phi=\int\limits_0^{\pi/2} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi=\int\limits_0^{\pi/2} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)^2\cos\phi\, d\phi$ Put $\sin\phi=t \Rightarrow \cos\phi\, d\phi=dt$ When $\phi=0, t=0$ and when $\phi=\frac{\pi}{2}, t=1 \therefore I=\int\limits_0^1 \sqrt{t}\left(1-t^2\right)^2 dt=\int\limits_0^1 \sqrt{t}\left(1-2t^2+t^4\right)dt=\int\limits_0^1 \left(t^{1/2}+t^{9/2}-2t^{5/2}\right)dt$
$=\left[\frac{2}{3}t^{3/2}+\frac{2}{11}t^{11/2}-\frac{4}{7}t^{7/2}\right]_0^1=\frac{2}{3}+\frac{2}{11}-\frac{4}{7}=\frac{154+42-132}{3\times 11\times 7}=\frac{64}{231}$

3. $\int\limits_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

SOLUTION

Let $I=\int\limits_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=\int\limits_0^{\pi/2} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)^2\cos\phi\, d\phi$ Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$ When $x=0, \theta=0$ and when $x=1, \theta=\frac{\pi}{4} \therefore I=\int\limits_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta=\int\limits_0^{\pi/4} \sin^{-1}(\sin 2\theta).\sec^2\theta d\theta=\int\limits_0^{\pi/4} 2\theta\sec^2\theta d\theta=\left[2\theta\tan\theta-2\int\left(\frac{d}{d\theta}(\theta)\cdot\tan\theta\right)d\theta\right]_0^{\pi/4}=$
$[2\theta\tan\theta-2\log\sec\theta]_0^{\pi/4}=\left(2\cdot\frac{\pi}{4}\cdot\tan\frac{\pi}{4}-2\log\sec\frac{\pi}{4}\right)-(0-2\log 1)=\frac{\pi}{2}-2\log\sqrt{2}=\frac{\pi}{2}-\log 2$

4. $\int\limits_0^2 x\sqrt{x+2}dx$

SOLUTION

: Let $I=\int\limits_0^2 x\sqrt{x+2}dx$ Put $x+2=t \Rightarrow dx=dt$ When $x=0, t=2$ and when $x=2, t=4 \therefore I=\int\limits_2^4 (t-2)\sqrt{t}dt=\int\limits_2^4 \left(t^{3/2}-2t^{1/2}\right)dt=$
$\left[\frac{2}{5}t^{5/2}-2\times\frac{2}{3}t^{3/2}\right]_2^4=\left[\frac{2}{5}t^{5/2}-\frac{4}{3}t^{3/2}\right]_2^4=\left[\frac{2}{5}(4)^{5/2}-\frac{4}{3}(4)^{3/2}\right]-\left[\frac{2}{5}(2)^{5/2}-\frac{4}{3}(2)^{3/2}\right]=\frac{2}{5}(2)^5-\frac{4}{3}(2)^3-\frac{2}{5}\times 4\sqrt{2}+\frac{4}{3}\times 2\sqrt{2}$
$=\frac{2}{5}\times 32-\frac{4}{3}\times 8-\frac{8}{2}\sqrt{2}+\frac{8}{3}\sqrt{2}=\frac{64}{5}-\frac{32}{3}-\left(\frac{8}{5}\sqrt{2}-\frac{8}{3}\sqrt{2}\right)=\frac{192-160}{15}-\left(\frac{24\sqrt{2}-40\sqrt{2}}{15}\right)=\frac{32}{15}+\frac{16\sqrt{2}}{15}=\frac{16}{15}\left(2+\sqrt{2}\right)=$
$\frac{16\sqrt{2}}{15}\left(\sqrt{2}+1\right)$

5. $\displaystyle\int_{0}^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

SOLUTION

: Let$I = \displaystyle\int_{0}^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$ Put $\cos x = t \Rightarrow -\sin x dx = dt$ When $x = 0, t = \cos 0 = 1$ and when $x = \dfrac{\pi}{2}, t = \cos\dfrac{\pi}{2} = 0 \therefore I = \displaystyle\int_{1}^{0} \frac{-dt}{1+t^2} =$
$-\left[\tan^{-1} t\right]_1^0 = \left[-\tan^{-1} 0 + \tan^{-1} 1\right] = \left[0 + \dfrac{\pi}{4}\right] = \dfrac{\pi}{4}$

6. $\displaystyle\int_{0}^{2} \frac{dx}{x+4-x^2}$

SOLUTION

Let $I = \displaystyle\int_{0}^{2} \frac{dx}{x+4-x^2} = \int_{0}^{2} \frac{dx}{4-(x^2-x)} = \int_{0}^{2} \frac{dx}{4+\frac{1}{4}-\left(x^2-x+\frac{1}{4}\right)} = \int_{0}^{2} \frac{dx}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2} = \int_{0}^{2} \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \frac{1}{2\cdot\frac{\sqrt{17}}{2}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}}{\frac{\sqrt{17}}{2}}\right.\right.$
$\dfrac{1}{\sqrt{17}}\left[\log\left(\dfrac{\sqrt{17}+2x-1}{\sqrt{17}-2x+1}\right)\right]_0^2 = \dfrac{1}{\sqrt{17}}\left[\log\left(\dfrac{\sqrt{17}+4-1}{\sqrt{17}-4+1}\right) - \log\left(\dfrac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right] = \dfrac{1}{\sqrt{17}}\left[\log\left(\dfrac{\sqrt{17}+3}{\sqrt{17}-3}\right) - \log\left(\dfrac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right] =$
$\dfrac{1}{\sqrt{17}}\log\left[\left(\dfrac{\sqrt{17}+3}{\sqrt{17}-3}\right)\left(\dfrac{\sqrt{17}+1}{\sqrt{17}-1}\right)\right] = \dfrac{1}{\sqrt{17}}\log\left[\dfrac{17+3+3\sqrt{17}+\sqrt{17}}{17+3-3\sqrt{17}-\sqrt{17}}\right] = \dfrac{1}{\sqrt{17}}\log\left(\dfrac{20+4\sqrt{17}}{20-4\sqrt{17}}\right) = \dfrac{1}{\sqrt{17}}\log\left(\dfrac{5+\sqrt{17}}{5-\sqrt{17}}\right) =$
$\dfrac{1}{\sqrt{17}}\log\left[\dfrac{5+\sqrt{17}}{5-\sqrt{17}} \times \dfrac{5+\sqrt{17}}{5+\sqrt{17}}\right] = \dfrac{1}{\sqrt{17}}\log\left[\dfrac{25+17+10\sqrt{17}}{8}\right] = \dfrac{1}{\sqrt{17}}\log\left[\dfrac{42+10\sqrt{17}}{8}\right] = \dfrac{1}{\sqrt{17}}\log\left[\dfrac{21+5\sqrt{17}}{4}\right]$

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7. $\displaystyle\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

SOLUTION

Let $I=\displaystyle\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{x^2+2x+1+4}=\int_{-1}^{1}\frac{dx}{(x+1)^2+2^2}=\frac{1}{2}\left[\tan^{-1}\frac{x+1}{2}\right]_{-1}^{1}=\frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]=\frac{1}{2}\times\frac{\pi}{4}=\frac{\pi}{8}$

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8. $\displaystyle\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

SOLUTION

: Let $I=\displaystyle\int_{1}^{2}e^{2x}\left(\frac{1}{x}-\frac{1}{2x^2}\right)dx$ Put $2x=t\Rightarrow 2dx=dt$ When $x=1, t=2$ and when $x=2, t=4$ $\therefore I=\displaystyle\frac{1}{2}\int_{2}^{4}e^{t}\left(\frac{2}{t}-\frac{1\times 4}{2t^2}\right)dt=$ $\displaystyle\frac{1}{2}\int_{2}^{4}e^{t}\left(\frac{2}{t}-\frac{2}{t^2}\right)dt=\int_{2}^{4}e^{t}\cdot\left(\frac{1}{t}-\frac{1}{t^2}\right)dt=\int_{2}^{4}e^{t}\cdot\left[\frac{1}{t}+\frac{d}{dt}\left(\frac{1}{t}\right)\right]dt=\left[e^{t}\cdot\frac{1}{t}\right]_{2}^{4}=\frac{1}{4}e^{4}-\frac{e^2}{2}=\frac{e^2}{2}\left(\frac{e^2}{2}-1\right)=\frac{e^2\left(e^2-2\right)}{4}$

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Choose the correct answer in Exercises 9 and 10. :

9. The value of the integral $\displaystyle\int_{1/3}^{1}\frac{\left(x-x^3\right)^{1/3}}{x^4}dx$ is

(a) 6
(b) 0
(c) 3
(d) 4

SOLUTION

(A): Let $I=\displaystyle\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ When $x=\frac{1}{3}, \sin\theta=\frac{1}{3}\Rightarrow\theta=\sin^{-1}\frac{1}{3}$ and When $x=1, \sin\theta=1\Rightarrow\theta=\frac{\pi}{2}$ $\therefore I=\displaystyle\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$

$=\displaystyle\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\sin^{\frac{1}{3}}\theta\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta\cos^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\sin^{\frac{1}{3}}\theta\cos^{\frac{5}{2}}\theta}{\sin^2\theta\sin^2\theta}d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\cos^{\frac{5}{3}}\theta}{\sin^{\frac{5}{3}}\theta}\cos ec^2\theta d\theta$ When

$\theta=\sin^{-1}\frac{1}{3}\Rightarrow\sin\theta=\frac{1}{3}\Rightarrow\cot\theta=2\sqrt{2}\Rightarrow t=2\sqrt{2}=\sqrt{8}$ and When $\theta=\frac{\pi}{2}, \cot\theta=0\Rightarrow t=0$ $\therefore I=\displaystyle\int_{\sqrt{8}}^{0}t^{\frac{5}{3}}(-dt)-=\int_{0}^{\sqrt{8}}t^{\frac{5}{3}}dt$

$=\displaystyle\left[\frac{t^{\frac{8}{3}}}{\frac{8}{3}}\right]_{0}^{\sqrt{8}}=\frac{3}{8}\left[\sqrt{8}\right]^{\frac{8}{3}}=\frac{3}{8}(8)^{\frac{8}{6}}=\frac{3}{8}(8)^{\frac{4}{3}}=\frac{3}{8}(16)=6$

10. If $f(x)=\displaystyle\int_{0}^{x}t\sin t\,dt$,then $f'(x)$ is

(a) $\cos x + x\sin x$

(b) $x\sin x$

(c) $x\cos x$

(d) $\sin x + x\cos x$

SOLUTION

(B): Let $f(x) = \int_0^x t\sin t\,dt \Rightarrow f(x) = \left[t(-\cos t) - \int 1\cdot(-\cos t)\,dt\right]_0^x = [-t\cos t + \sin t]_0^x \Rightarrow f(x) = -x\cos x + \sin x \therefore f'(x) = -[x(-\sin x) + \cos x] + \cos x = x\sin x - \cos x + \cos x = x\sin x$