NCERT - Miscellaneous Exercise

Differentiate w.r.tx the function in exercises 1 to 11.

1. $(3x^2-9x+5)^9$

SOLUTION

Let$y=(3x^2-9x+5)^9$ ...(i) Differentiating (i) w.r.t. x, we get $\frac{dy}{dx}=9(3x^2-9x+5)^8\cdot(6x-9)=27(3x^2-9x+5)^8\cdot(2x-3)$

2. $\sin^3 x+\cos^6 x$

SOLUTION

Let$y=\sin^3 x+\cos^6 x\ldots$(i) Differentiating (i) w.r.t. x, we get $\frac{dy}{dx}=3\sin^2 x\cos x+6\cos^5 x(-\sin x)=3\sin x\cos x(\sin x-2\cos^4 x)$

3. $(5x)^{3\cos 2x}$

SOLUTION

Let $y=(5x)^{3\cos 2x}$ Taking log on both sides, we get

$\log y=3\cos 2x\log(5x)=3\cos 2x[\log 5+\log x]$ $\log y=3\cos 2x\log 5+3\cos 2x\log x\ldots$(i)

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=3\log 5(-\sin 2x)\cdot 2+\frac{3\cos 2x}{x}-3\log x\cdot(2\cdot\sin 2x)$$

$$=-6\log 5\sin 2x+\frac{3\cos 2x}{x}-6\log x\sin 2x$$

$$\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6[\log 5+\log x]\sin 2x\right]$$

$$=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\log 5x\sin 2x\right]$$

4. $\sin^{-1}(x\sqrt{x}), 0\le x\le 1.$

SOLUTION

Let y= $\sin^{-1}(x\sqrt{x})\ldots$(i) Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}x\sqrt{x}=\frac{1}{\sqrt{1-x^3}}\left[x\cdot\frac{1}{2\sqrt{x}}+\sqrt{x}\right]$$

$$=\frac{1}{\sqrt{1-x^3}}\left[\frac{\sqrt{x}}{2}+\sqrt{x}\right]=\frac{1}{\sqrt{1-x^3}}\left[\frac{\sqrt{x}+2\sqrt{x}}{2}\right]=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

5. $\frac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{2x+7}}, -2<x<2.$

SOLUTION

Let $y=\cos^{-1}\frac{x}{2}(2x+7)^{-1/2}$ ..(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\cos^{-1}\frac{x}{2}\left[\frac{d}{dx}(2x+7)^{-1/2}\right]+(2x+7)^{-1/2}\left(\frac{d}{dx}\cos^{-1}\frac{x}{2}\right)$$

$$= \cos^{-1}\frac{x}{2}\left[\frac{-1}{2}(2x+7)^{-3/2}(2)\right] + (2x+7)^{-1/2}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\times\frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = -\cos^{-1}\frac{x}{2}(2x+7)^{-3/2} + (2x+7)^{-1/2}\left[\frac{-1}{2\sqrt{1-\left(\frac{x^2}{4}\right)}}\right]$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{3/2}}\right]$$

6. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}.$

SOLUTION

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

$$\Rightarrow y = \cos^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\cot\frac{x}{2}\right] y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

7. $(\log x)^{\log x}, x > 1$

SOLUTION

Let $y = (\log x)^{\log x}$ Taking log on both sides, we get log y = log x log (log x) ...(i) Differentiating (i) on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log x \cdot \frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\frac{1}{x} = \frac{1}{x}\cdot[1+\log(\log x)]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\cdot\frac{1}{x}\cdot[1+\log(\log x)].$$

8. cos(a cos x + b sin x), for some constant a and b.

SOLUTION

Let y = cos (a cos x + b sin x) ...(i)

Differentiating (i) on both sides w.r.t. x, we get $\frac{dy}{dx} = -$ sin (a cos x + b sin x) [a (−sin x) + b cos x]

$= -$ sin (a cos x + b sin x) [−a sin x + b cos x] = (a sin x− b cos x) sin (a cos x + b sin x)

9. $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$.

SOLUTION

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$ Taking log on both sides, we get log y = (sin x − cos x) log (sin x − cos x) ...(i)

Differentiating (i) on both sides w.r.t. x, we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x) \cdot (\cos x + \sin x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + \log(\sin x - \cos x) \cdot (\cos x + \sin x)$

$\Rightarrow \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$

$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \ [(\cos x + \sin x)(1 + \log(\sin x - \cos x))], \sin x > \cos x$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

SOLUTION

Let $y = x^x(I) + x^a(II) + a^x(III) + a^a(III)$

I. Let $y = x^x \Rightarrow \log y = \log x^x$

$\Rightarrow \log y = x \log x$ ..(i) Differentiating (i) on both sides w.r.t. x, we get

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x(i) \Rightarrow \frac{dy}{dx} = y[1 + \log x]$

$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x) \ \ldots$(ii)

II. Let $y = x^a \Rightarrow \log y = \log x^a \Rightarrow \log y = a \log x \ldots$(iii) Differentiating (iii) on both sides w.r.t. x, we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a\frac{1}{x} \Rightarrow \frac{dy}{dx} = y\left(\frac{a}{x}\right)$

$\Rightarrow \frac{dy}{dx} = x^a\left(\frac{a}{x}\right) = x^a \cdot a \cdot x^{-1} = ax^{a-1} \ \ldots$(iv)

III. Let $y = a^x \Rightarrow \log y = \log a^x$

$\Rightarrow \log y = x \log a \ldots$(v)

Differentiating (v) on both sides w.r.t. x, we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log a(1) \Rightarrow \frac{dy}{dx} = y \log a = a^x \log a \ldots$(vi) IV. Let $y = a^a \Rightarrow \log y = \log a^a$

$\Rightarrow \log y = a \log a$

Differentiating (vii) on both sides w.r.t. x, we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a(0) \Rightarrow \frac{dy}{dx} = 0 \ldots$(viii)

$\therefore \frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a$ [from (ii), (iv), (vi) and (viii)]

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11. $x^{x^2-3}+(x-3)^{x^2}$, for $x>3$.

SOLUTION

Let $y=x^{x^2-3}+(x-3)^{x^2}=u+v$, where $u=x^{x^2-3}$, and $v=(v-3)^{x^2}$

Let $u=x^{x^2-3}$

Taking log on both sides, we get $\log u=(x^2-3)\log x$ ...(i)

Differentiating (i) w.r.t. x on both sides, we get $\frac{1}{u}\frac{du}{dx}=\frac{(x^2-3)}{x}+\log x(2x)$

$\therefore \frac{du}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]\ldots$(ii)

and $v=(x-3)^{x^2}$

Taking log on both sides, we get $\log v=x^2\log(x-3)\ldots$.(iii)

Differentiating (iii) w.r.t.x, we get

$\frac{1}{v}\frac{dv}{dx}=\frac{x^2}{x-3}+\log(x-3)\cdot(2x)$

$\therefore \frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]\ldots$.(iv)

So, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

$=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$

[from (ii) & (iv)]

12. Find $\frac{dy}{dx}, if y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$.

SOLUTION

Here, $y=12(1-\cos t)\ldots$.(i) $x=10(t-\sin t)\ldots$(ii)

Differentiating (1) (2) w.r.t. t, we get $\frac{dy}{dt}=12[-(-\sin t)]=12\sin t$

and $\frac{dx}{dt}=10(1-\cos t)$

$\Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{12\sin t}{10(1-\cos t)}=\frac{6\sin t}{5(1-\cos t)}$

$=\frac{6}{5}\left[\frac{2\sin(t/2)\cos(t/2)}{2\sin^2(t/2)}\right]=\frac{6}{5}\cot(t/2)$

13. Find, $\frac{dy}{dx}, if y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}, -1\leq x\leq 1$.

SOLUTION

Here, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}=u+v$ Let $u=\sin^{-1}x$ and $v=\sin^{-1}\sqrt{1-x^2}$

$\Rightarrow \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$ and for $v=\sin^{-1}\sqrt{1-x^2}$ Putting $x=\cos\theta$, we get

$v=\sin^{-1}\sqrt{1-\cos^2\theta}=\sin^{-1}\sqrt{\sin^2\theta}=\sin^{-1}(\sin\theta)=\theta=\cos^{-1}x$

$\therefore \frac{dv}{dx}=\frac{-1}{\sqrt{1-x^2}}$

So, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{-1}{\sqrt{1-x^2}}=0$

14. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$, prove that $\dfrac{dy}{dx}=-\dfrac{1}{(1+x)^2}$

SOLUTION

we have, $x\sqrt{1+y}+y\sqrt{1+x}=0 \Rightarrow x\sqrt{1+y}=-y\sqrt{1+x} \Rightarrow x^2(1+y)=y^2(1+x)$

$\Rightarrow (x^2-y^2)+xy(x-y)=0 \Rightarrow y=\dfrac{-x}{x+1}$

$\therefore \dfrac{dy}{dx}=\dfrac{(x+1)(-1)-(-x)(1)}{(x+1)^2}=\dfrac{-x-1+x}{(x+1)^2}=\dfrac{-1}{(x+1)^2}$

15. If $(x-a)^2+(y-b)^2=c^2$, for some $c>0$, prove that

$$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

is a constant independent of a and b.

SOLUTION

We have, $(x-a)^2+(y-b)^2=c^2 \ldots$(i) Differentiating (i) with respect to x, we get $2(x-a)+2(y-b)\dfrac{dy}{dx}=0$

$\Rightarrow (x-a)+(y-b)\dfrac{dy}{dx}=0$

Differentiating (ii) with respect to x, we get $1+(y-b)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$

$\Rightarrow (y-b)\dfrac{d^2y}{dx^2}=-\left[1+\left(\dfrac{dy}{dx}\right)^2\right]$

$\Rightarrow \dfrac{d^2y}{dx^2}=-\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{(y-b)} \ldots$(iii)

From (ii), we have $\dfrac{dy}{dx}=-\left(\dfrac{x-a}{y-b}\right) \Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2=\left(-\left(\dfrac{x-a}{y-b}\right)\right)^2$ Adding 1 on both sides, we get

$1+\left(\dfrac{dy}{dx}\right)^2=1+\left(-\left(\dfrac{x-a}{y-b}\right)\right)^2=1+\left(\dfrac{x-a}{y-b}\right)^2=\dfrac{c^2}{(y-b)^2} \ldots$(iv)

Also, $\dfrac{d^2y}{dx^2}=-\dfrac{c^2}{(y-b)^3} \ldots$(v)

[From (iii) & (iv)] From (iv) and (v), we have

$$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}=\frac{\dfrac{c^3}{(y-b)^3}}{\dfrac{-c^2}{(y-b)^3}}=-c$$

which is independent of a and b.

16. If cos y=x cos(a + y), with cos a$\neq \pm$ 1, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$.

SOLUTION

We have, cos y =x cos (a + y) $\Rightarrow x=\frac{\cos y}{\cos(a+y)}$

$\Rightarrow \frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}$

$=\frac{\cos y\sin(a+y)-\sin y\cos(a+y)}{\cos^2(a+y)}$

$=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}\therefore \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

17. If $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$.

SOLUTION

Here, $\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)\therefore \frac{dx}{dt}=at\cos t$

$\Rightarrow \frac{dy}{dt}=a[\cos t-\{t(-\sin t)+\cos t(1)\}]$

$\Rightarrow \frac{dy}{dt}=a(\cos t+t\sin t-\cos t)=at\sin t$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=at\sin t\times\frac{1}{at\cos t}=\tan t$

$\Rightarrow \frac{d^2y}{dx^2}=\sec^2 t\cdot\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=\sec^2 t\cdot\frac{1}{at\cos t}=\frac{\sec^2 t}{at\cos t}=\frac{\sec^3 t}{at}, 0<t<\frac{\pi}{2}$

18. If $f(x)=|x|^3$, show that fix)existsfor all real x and find it

. SOLUTION

Case I. When $x\geq 0$. Here,$f(x)=|x|^3=x^3$

$\therefore f'(x)=3x^2$ and Case II. When x < 0. Here $f(x)=(-x)^3=-x^3$

$\therefore f'(x)=-3x^2$ and Thus, Hence, .

19. Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers it.

SOLUTION

Let $P(n)$ be the given statement in the problem

$P(n):\frac{d}{dx}(x^n)=nx^{n-1}\ldots$(i) For n = 1, Putting n = 1 in (i), we get

$P(1):\frac{d}{dx}(x^1)=(1)x^{1-1}=(1)x^0=(1)(1)=1$ which is true as (x) = 1

We suppose $P(n)$ is true for n = m, $P(m):\frac{d}{dx}(x^m)=mx^{m-1}\ldots$(ii) To establish the truth of $P(m+1)$, we prove

$P(m+1):\frac{d}{dx}(x^{m+1})=(m+1)x^m\ldots$(ii)

Now, $x^{m+1}=x^1\cdot x^m$

$\Rightarrow \frac{d}{dx}(x^{m+1})=\frac{d}{dx}(x\cdot x^m)=x\cdot\frac{d}{dx}(x^m)+x^m\frac{d}{dx}(x)$

$=x\cdot mx^{m-1}+x^m=mx^m+x^m=x^m(m+1)=(m+1)x^{(m+1)-1}$

$\therefore P(n)$ is true for n= m+ 1. By principle of induction, P(n) is true for all $n\in N$.

20. Using the fact that $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and the differentiation, obtain the sum formula for cosines.

SOLUTION

s $\sin(A+B)=\sin A\cos B+\cos A\sin B$ ....(i)

Consider A and B as function of t and differentiating both sides of (i) w.r.t. t, we get $\cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)$

$$=\sin A(-\sin B)\frac{dB}{dt}+\cos B\left[\cos A\frac{dA}{dt}\right]+\cos A\cos B\frac{dB}{dt}+\sin B(-\sin A)\frac{dA}{dt}$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)$$

$$=(\cos A\cos B-\sin A\sin B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)$$

$\Rightarrow\cos(A+B)=\cos A\cos B-\sin A\sin B$.

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

SOLUTION

Let the function be $f(x)=|x-1|+|x-2|$. We redefine f(x) as : $f(x)=\begin{cases}-(x-1)-(x-2); & if \quad x<1\\ ((x-1)-(x-2); & if \quad 1\le x\le 2\\ (x-1)+(x-2); & if \quad x>2\end{cases}$

$$\Rightarrow f(x)=\begin{cases}-2x+3; & if \quad x<1\\ 1; & if \quad 1\le x\le 2\\ (2x-3); & if \quad x>2\end{cases}$$

$f(x)$ is clearly continuous at all x except at x=1, 2. At x= 1:

$\lim\limits_{x\to1^-}f(x)=\lim\limits_{\substack{x\to1-h\\h\to0}}(-2(1-h)+3)=-2+3=1$ $\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}(1)=1.$

Also, $f(1)=1$

Thus, $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^+}f(x)=f(1)$

Hence, $f(x)$ is continuous at x= 1.

At x $=2:$ $\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^-}1=1$

$\lim\limits_{x\to2^+}f(x)=\lim\limits_{x\to2^+}(2x-3)=\lim\limits_{\substack{x\to2+h\\h\to0}}(2(2+h)-3)=2(2)-3=1$

Also, $f(2)=1$

Thus, $\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=f(2)$

Hence, f(x) is continuous at x $=2$. Hence, f(x) is continuous at all $x\in R$.

Derivability at x=1: $Lf'(1)=\lim\limits_{h\to0}\frac{f(1-h)-f(1)}{-h}$

$=\lim\limits_{h\to0}\frac{-2(1-h)+3-1}{-h}=\lim\limits_{h\to0}\frac{2h}{-h}=\lim\limits_{h\to0}(-2)=-2$

$Rf'(1)=\lim\limits_{h\to0}\frac{f(1+h)-f(1)}{h}=\lim\limits_{h\to0}\frac{1-1}{h}=0$

Thus, $Lf'(1)\neq Rf'(1)\Rightarrow$ f is not derivable at x=1 Derivability at x=2 :

$Lf'(2)=\lim\limits_{h\to0}\frac{f(2-h)-f(2)}{-h}=\lim\limits_{h\to0}\frac{(1)-(1)}{-h}=0$

$Rf'(2)=\lim\limits_{h\to0}\frac{f(2+h)-f(2)}{h}$

$=\lim\limits_{h\to0}\frac{2(2+h)-3-1}{h}=\lim\limits_{h\to0}\frac{2h}{h}=\lim\limits_{h\to0}2=2$

Thus, $Lf'(2)\neq Rf'(2)\Rightarrow$ f is not derivable at x= 2

Hence $f(x)=|x-1|+|x-2|$ is continuous everywhere and differentiable at all x e except at 1 and 2.

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

SOLUTION

We have , $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Hence proved.

23. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

SOLUTION

We have, $y = e^{a\cos^{-1}x} \ldots$(i)

Differentiating (i) on both sides w.r.t. x, we get $\frac{dy}{dx} = e^{a\cos^{-1}x}\frac{d}{dx}(a\cos^{-1}x)$

$$= e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = \frac{-ay}{\sqrt{1-x^2}} \ldots \text{(ii)}$$

Differentiating (ii) on both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - y\frac{d}{dx}\sqrt{1-x^2}}{(1-x^2)}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - \frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x)^2}\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + \frac{xy}{\sqrt{1-x^2}}\right] \text{ [From (ii)]}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + x\cdot\left(\frac{-1}{a}\cdot\frac{dy}{dx}\right)\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$