NCERT - Exercise 5.7

Find the second order derivatives of the functions given in questions 1 to 10.

1. x^2+3x+2

SOLUTION

Let y = x2 + 3x + 2 $\Rightarrow \frac{dy}{dx}=2x+3 \therefore \frac{d^2y}{dx^2}=2$

2. x^{20}

SOLUTION

$\text{Let} y=x^{20} \Rightarrow \frac{dy}{dx}=20x^{19} \therefore$

$\frac{d^2y}{dx^2}=20\cdot 19x^{18}=380x^{18}$

3. $x\cdot\cos x$

SOLUTION

Let y = xcos x $\Rightarrow \frac{dy}{dx}=x\cdot(-\sin x)+\cos x=-x\sin x+\cos x$

$\Rightarrow \frac{d^2y}{dx^2}=-[x\cos x+\sin x]-\sin x=-(x\cos x+2\sin x)$

4. log x

SOLUTION

Let y = log x $\Rightarrow \frac{dy}{dx}=\frac{1}{x}\therefore$

$\frac{d^2y}{dx^2}=-\frac{1}{x^2}$

5. $x^3\log x$

SOLUTION

Let $y=x^3\log x$

$\Rightarrow \frac{dy}{dx}=x^3\cdot\frac{1}{x}+\log x\cdot 3x^2=x^2+3x^2\log x$

$\Rightarrow \frac{d^2y}{dx^2}=2x+3\left[x^2\cdot\frac{1}{x}+\log x\cdot 2x\right]$

$=2x+3[x+2x\log x]=2x+3x+6x\log x=5x+6x\log x=x(5+6\log x)$

6. $e^x\sin 5x$

SOLUTION

Let $y=e^x\sin 5x \Rightarrow \frac{dy}{dx}=e^x\cdot\cos 5x\cdot 5+\sin 5x\cdot e^x$

$=e^x[5\cos 5x+\sin 5x]\therefore \frac{d^2y}{dx^2}=e^x[5(-\sin 5x)\cdot 5+\cos 5x\cdot 5]+[5\cos 5x+\sin 5x]e^x$

$=e^x[-25\sin 5x+5\cos 5x+5\cos 5x+\sin 5x]=e^x[10\cos 5x-24\sin 5x]=2e^x[5\cos 5x-12\sin 5x]$

7. $e^{6x}\cos 3x$

SOLUTION

$$\text{Let} y = e^{6x}\cos 3x \Rightarrow \frac{dy}{dx} = e^{6x}(-\sin 3x)\cdot 3 + \cos 3x \cdot e^{6x}\cdot 6 = 6e^{6x}\cos 3x - 3e^{6x}\sin 3x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6[e^{6x}(-\sin 3x)\cdot 3 + \cos 3x \cdot e^{6x}.6] - 3[e^{6x}\cos 3x \cdot 3 + \sin 3x e^{6x}\cdot 6]$$

$$= -18e^{6x}\sin 3x + 36e^{6x}\cos 3x - 9e^{6x}\cos 3x - 18e^{6x}\sin 3x = 27e^{6x}\cos 3x - 36e^{6x}\sin 3x = 9e^{6x}(3\cos 3x - 4\sin 3x)$$

8. $\tan^{-1}x$

SOLUTION

$$\text{Let } y = \tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{1}{(1+x^2)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1+x^2)\cdot 0 - 1\cdot(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

9. $\log(\log x)$

SOLUTION

$$\text{Let} y = \log(\log x) \Rightarrow \frac{dy}{dx} = \frac{1}{\log x}\frac{1}{x}\therefore$$

$$\frac{d^2y}{dx^2} = \frac{1}{\log x}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right)$$

$$= \frac{-1}{x^2\log x} + \frac{1}{x}\left[\frac{\log x\cdot 0 - 1\cdot\frac{1}{x}}{(\log x)^2}\right] = \frac{-1}{x^2\log x} + \frac{1}{x}\left[\frac{-\frac{1}{x}}{(\log x)^2}\right]$$

$$= \frac{-1}{x^2\log x} - \frac{1}{x^2(\log x)^2} = \frac{-1}{x^2\log x}\left[1 + \frac{1}{\log x}\right] = \frac{-(1+\log x)}{(x\log x)^2}$$

10. $\sin(\log x)$

SOLUTION

$$\text{Let y = sin (log x)} \Rightarrow \frac{dy}{dx} = \cos(\log x)\frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \cos(\log x)\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\{-\sin(\log x)\}\cdot\frac{1}{x}$$

$$= \frac{-\cos(\log x)}{x^2} - \frac{\sin(\log x)}{x^2}$$

$$= -\frac{1}{x^2}[\cos(\log x) + \sin(\log x)]$$

11. If $y = 5\cos x - 3\sin x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

SOLUTION

We have, $y = 5\cos x - 3\sin x$

$$\Rightarrow \frac{dy}{dx} = 5(-\sin x) - 3(\cos x) = -5\sin x - 3\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -5\cos x - 3(-\sin x) = -5\cos x + 3\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = -5\cos x + 3\sin x + 5\cos x - 3\sin x = 0 \text{ Hence proved.}$$

12. If $y=\cos^{-1}x$, find $\dfrac{d^2y}{dx^2}$in terms ofy alone.

SOLUTION

$y=\cos^{-1}x \Rightarrow x=\cos y \ldots$(i) Differentiating (i) w.r.t. x, we get $1=-\sin y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}=-\cos ec y \ldots$(ii)

Differentiating (ii) w.r.t. x, we get $\dfrac{d^2y}{dx^2}=\cos ec y \cot y\dfrac{dy}{dx}=-\cos ec^2 y \cot y$

13. If $y=3\cos(\log x)+4\sin(\log x)$, then show that $x^2y_2+xy_1+y=0$

SOLUTION

We have, $y=3\cos(\log x)+4\sin(\log x), \ldots.$(i) Differentiating (i) w.r.t. x, we get

$\dfrac{dy}{dx}=3[-\sin(\log x)]\dfrac{1}{x}+4[\cos(\log x)]\dfrac{1}{x}\ldots$(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{d^2x}=3\left[\left(-\sin(\log x)\left(-\frac{1}{x^2}\right)\right)+\frac{1}{x}(-\cos(\log x))\frac{1}{x}\right]+4\left[\cos(\log x)\cdot\left(\frac{-1}{x^2}\right)+\frac{1}{x}\{-\sin(\log x)\}\frac{1}{x}\right]$$

$=\dfrac{1}{x^2}[3\sin(\log x)-3\cos(\log x)-4\cos(\log x)-4\sin(\log x)]$

$\Rightarrow x^2\dfrac{d^2y}{dx^2}=[3\sin(\log x)-4\cos(\log x)-\{3\cos(\log x)+4\sin(\log x)\}$

$\Rightarrow x^2\dfrac{d^2y}{dx^2}=-x\dfrac{dy}{dx}-y$

$\Rightarrow x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0 \Rightarrow x^2y_2+xy_1+y=0$ Hence proved.

14. If$y=Ae^{mx}+Be^{nx}$,then show that $\dfrac{d^2y}{dx^2}-(m+n)\dfrac{dy}{dx}+mny=0$.

SOLUTION

Let$y=Ae^{mx}+Be^{nx} \ldots$(i) Differentiating (i) w.r.t. x,we get

$\dfrac{dy}{dx}=Ae^{mx}\cdot m+Be^{nx}\cdot n=Ame^{mx}+Bne^{nx}\ldots$(ii)

Differentiating (ii) w.r.t.x, we get

$\dfrac{d^2y}{dx^2}=Ame^{mx}\cdot m+Bne^{nx}\cdot n=Am^2e^{\mathrm{mx}}+Bn^2e^{nx}\ldots$(iii)

Now, $\dfrac{d^2y}{dx^2}-(m+n)\dfrac{dy}{dx}+mny=Am^2e^{\mathrm{mx}}+Bn^2e^{nx}$

$-[(m+n)(Ame^{mx}+Bne^{nx})]+mn(Ae^{\mathrm{mx}}+Be^{nx})$ [from (i),(ii) and (iii)]

$=Am^2e^{mx}+Bn^2e^{nx}-Am^2e^{mx}-Bmne^{nx}-Amne^{\mathrm{mx}}-Bn^2e^{nx}+Amne^{mx}+Bmne^{nx}=0$ Hence proved.

15. If $y=500e^{7x}+600e^{-7x}$, then show that $\dfrac{d^2y}{dx^2}=49y$.

SOLUTION

Let $y=500e^{7x}+600e^{-7x}$ ....(i) Differentiating (i) w.r.t. x, we get

$\dfrac{dy}{dx}=500\cdot e^{7x}\cdot 7+600\cdot e^{-7x}\cdot(-7)$

$=3500e^{7x}-4200e^{-7x}\ldots$(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 3500 \cdot 7 \cdot e^{7x} - 4200 \cdot (-7)e^{-7x} = 24500e^{7x} + 29400e^{-7x}$$

$$= 49(500e^{7x} + 600e^{-7x}) = 49y \therefore \frac{d^2y}{dx^2} = 49y$$

16. If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

SOLUTION

$xe^y + e^y = 1$ ...(i) Differentiating (i) w.r.t. x, we get

$$xe^y\frac{dy}{dx} + e^y + e^y\frac{dy}{dx} = 0 \Rightarrow$$

$$\frac{dy}{dx} = \frac{-1}{x+1} \dots \text{(ii)}$$

From (ii), $\left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2} \dots \text{(iii)}$

Differentiating (ii) w.r.t. x, we get $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$ ..(iv)

From (iii) and (iv), we get $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Hence proved. .

17. If $y = (\tan^{-1}x)^2$, show that $(x^2+1)^2y_2 + 2x(x^2+1)y_1 = 2$.

SOLUTION

Let $y = (\tan^{-1}x)^2$ ...(i) Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x \cdot \frac{1}{(1+x^2)} \dots \text{.(ii)}$$

Differentiating (ii) w.r.t.x, we get

$$\frac{d^2y}{dx^2} = 2\left[\tan^{-1}x\frac{(\{1+x^2\} \cdot 0 - 2x)}{(1+x^2)^2} + \frac{1}{(1+x)^2} \cdot \frac{1}{(1+x^2)}\right]$$

$$= 2\left[\frac{-2x\tan^{-1}x}{(1+x^2)^2} + \frac{1}{(1+x^2)^2}\right] = 2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right]$$

Now, $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx}$

$= (x^2+1)^2 \cdot 2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right] + 2x(x^2+1) \cdot 2\tan^{-1}x\frac{1}{(1+x^2)} = -4x\tan^{-1}x + 2 + 4x\tan^{-1}x = 2$ Hence proved.