NCERT - Exercise 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at x= 0, at $x = 0$ and at$x = 5$.

SOLUTION

$f(x) = 5x - 3$ At $x = 0$: We have, $f(0) = -3$

$\lim_{x \to 0^-} f(x)$

$= \lim_{\substack{x \to 0-h \\ h \to 0}} 5(0-h) - 3$

$= -3$

$\lim_{x \to 0^+} f(x) = \lim_{\substack{x \to 0+h \\ h \to 0}} 5(0+h) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ $\therefore$ f is continuous at x = 0.

At $x = -3$: We have,$f(-3) = 5(-3) - 3 = -18$

$\lim_{x \to -3^-} f(x) = \lim_{\substack{x \to 3-h \\ h \to 0}} [5(-3-h) - 3] = \lim_{\substack{x \to 3-h \\ h \to 0}} [-15 - 5h - 3] = -18$

$\lim_{x \to -3^+} f(x) = \lim_{\substack{x \to 3+h \\ h \to 0}} [5(-3+h) - 3] = \lim_{\substack{x \to 3+h \\ h \to 0}} -15 + 5h - 3] = -18$

$\therefore \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} f(x) = f(-3)$

$\therefore$ f is continuous at $x = -3$. .

At $x = 5$: $f(5) = 5(5) - 3 = 22$

$\lim_{x \to 5^-} f(x) = \lim_{\substack{x \to 5-h \\ h \to 0}} [5(5-h) - 3] = \lim_{\substack{x \to 5-h \\ h \to 0}} 25 - 5h - 3 = 22$

$\lim_{x \to 5^+} f(x) = \lim_{\substack{x \to 5+h \\ h \to 0}} [5(5+h) - 3] = \lim_{\substack{x \to 5+h \\ h \to 0}} 25 + 5h - 3 = 22$

$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

$\therefore$ f is continuous at $x = 5$.

2. Examine the continuity of the function f(x)$2x^2 - 1$ at $x = 3$.

SOLUTION

$f(x) = 2x^2 - 1; R.H.L.$

$= \lim_{x \to 3^+} f(x) = \lim_{\substack{x \to 3+h \\ h \to 0}} 2(3+h)^2 - 1$

$= \lim_{\substack{x \to 3+h \\ h \to 0}} 2(9 + 6h + h^2) - 1$

$= \lim_{\substack{x \to 3+h \\ h \to 0}} (18 + 12h + 2h^2) - 1 = \lim_{\substack{x \to 3+h \\ h \to 0}} (17 + 12h + 2h^2) = 17$

$L.H.L. = \lim_{x \to 3^-} f(x) = \lim_{\substack{x \to 3-h \\ h \to 0}} 2(3-h)^2 - 1$

$= \lim_{\substack{x \to 3-h \\ h \to 0}} 2(9 - 6h + h^2) - 1$

$= \lim_{\substack{x \to 3-h \\ h \to 0}} (18 - 12h + 2h^2) - 1$

$= \lim_{\substack{x \to 3-h \\ h \to 0}} 2h^2 - 12h + 17 = 17$

$\therefore R.H.L. = L.H.L.$

Also, $f(3)=2(3)^2-1=17 \therefore \lim_{x\to 3^+} f(x)=\lim_{x\to 3^-} f(x)=f(3)$

Hence, the given function $f(x)=2x^2-1$ is continuous at x = 3.

3. Examine the following functions for continuity :

(a)$f(x)=x-5$

(b)$f(x)=\dfrac{1}{x-5}, x\neq 5$

(c) $f(x)=\dfrac{x^2-25}{x+5}, x\neq -5$

(d) $f(x)=|x-5|$

SOLUTION

(a) $f(x)=x-5$

Let a be a real number, then

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\ h\to 0}}(a+h)-5=a-5$

$\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\ h\to 0}}(a-h)-5=a-5$

Also, $f(a)=a-5 \therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence, the given function $f(x)=(x-5)$ is continuous.

SOLUTION

(b) $f(x)=\dfrac{1}{x-5}$ Let a be a real number, then

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\ h\to 0}}\dfrac{1}{a+h-5}=\dfrac{1}{a-5}$

$\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\ h\to 0}}\dfrac{1}{a-h-5}=\dfrac{1}{a-5}$

Also, $f(a)=\dfrac{1}{a-5} \therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence, the given function $f(x)=\dfrac{1}{x-5}$ is continuous at all point except at $x=5$.

SOLUTION

(c)$f(x)=\dfrac{x^2-25}{x+5}=\dfrac{(x+5)(x-5)}{(x+5)}=x-5$ Let 'a' be a real number, then

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\ h\to 0}}(a+h)-5=a-5$

and $\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\ h\to 0}}(a-h)-5=a-5$

Also, $f(a)=a-5 \therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence, the given function $f(x)=x-5$ is continuous at every point of its domain.

SOLUTION

(d) $f(x)=|x-5|$

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\ h\to 0}}|a+h-5|=|a-5|=a-5$

$\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\ h\to 0}}|a-h-5|=|a-5|=a-5$

Also,$f(a)=|a-5|=a-5 \therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence, the given function $f(x)=|x-5|$ is continuous.

4. Prove that the function $f(x) = x^n$ is continuous at x = n, where n is a positive integer.

SOLUTION

Given, $f(x) = x^n, n \in N$ So, f (x) is a polynomial function and domain of f is R. $\lim_{x\to n} f(x) = \lim_{x\to n} x^n = x^n = f(n)$

$\Rightarrow$ f is continuous at n$\in$N.

5. Is the function f defined by $f(x) = \begin{cases} x, & if \quad x \leq 1 \\ 5, & if \quad x > 1 \end{cases}$ continuous at x = 0 ? At x = 1? At x = 2 ?

SOLUTION

(i) At x = 0, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x) = 0$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0$

Also, $f(0) = 0$

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Hence, f(x) is continuous at x = 0.

(ii) At x = 1, . $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$ $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5 = 5$

$\therefore \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

$\Rightarrow$ f is discontinuous at x = 1.

(iii) At x = 2, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 5 = 5$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5 = 5$

Also, f(2) = 5

Thus $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$ $\therefore$ f(x) is continuous at x = 2.

Direction : For questions (6 - 12), find all points of discontinuity of functionf(x). .

6. $f(x) = \begin{cases} 2x+3, & if \quad x \leq 2 \\ 2x-3, & if \quad x > 2 \end{cases}$

SOLUTION

For x < 2, function f (x)= 2x + 3 is polynomial and hence, continuous. For $x > 2$, function f(x) = 2x−3 is polynomial and hence, continuous.

For continuity at x = 2,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+3) = \lim_{\substack{x\to 2-h \\ h\to 0}} [2(2-h)+3]$

$= \lim_{\substack{x\to 2-h \\ h\to 0}} (4-2h+3) = \lim_{\substack{x\to 2-h \\ h\to 0}} (7-2h) = 7$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x-3) = \lim_{\substack{x\to 2+h \\ h\to 0}} [2(2+h)-3]$

$= \lim_{\substack{x\to 2+h \\ h\to 0}} (4+2h-3) = \lim_{\substack{x\to 2+h \\ h\to 0}} (1+2h) = 1$

Thus, $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ $\therefore$ f(x) is not continuous at x = 2.

So, the only point of discontinuity of f is 2.

7. $f(x) = \begin{cases} |x|+3, & if x \leq -3 \\ -2x, & if \ -3 < x < 3 \\ 6x+2, & if x \geq 3 \end{cases}$

SOLUTION

At x = 3 :

$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} |x|+3 = \lim_{\substack{x\to -3-h\\ h\to 0}} (|-3-h|+3) = |-3-0|+3 = 3+3 = 6$

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (-2x) = \lim_{\substack{x\to -3+h\\ h\to 0}} (-2(-3+h3)) = -2(-3+0) = 6$ $f(-3) = |-3|+3 = 3+3 = 6$

Thus,, $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = f(-3)$

$\therefore$ f is continuous at $x = -3$.

At x = 3 :

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x) = \lim_{\substack{x\to 3-h\\ h\to 0}} (-2(3-h)) = -2(3-0) = -6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x+2) = \lim_{\substack{x\to 3+h\\ h\to 0}} (6(3+h)+2) = 6(3+0)+2 = 20$

Thus, $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$ $\therefore f(x)$ is discontinuous at x = 3.

So, the only point of discontinuity of f is 3.

8. $f(x) = \begin{cases} \frac{|x|}{x}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$

SOLUTION

At x = 0 : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} (1) = 1$

Thus, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \Rightarrow$ f(x) is discontinuous at $x = 0$.

So, the only point of discontinuity of f is 0.

9. $f(x) = \begin{cases} \frac{x}{|x|}, if & x < 0 \\ -1, if & x \geq 0 \end{cases}$

SOLUTION

At x = 0 : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x}$

$\lim_{x\to 0^-} (-1) = -1$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-1) = -1$

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow f(x)$ is continuous at x = 0.

So, f(x) has no point of discontinuity.

10. . $f(x) = \begin{cases} x+1, & if x \geq 1 \\ x^2+1, & if x < 1 \end{cases}$

SOLUTION

We observe that f(x) is continuous at all real numbers $x < 1$ and $x > 1$ as it is polynomial function.

Now, continuity at x = 1 :

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = \lim_{x\to 1+h} (1+h)+1 = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+1) = \lim_{x\to 1-h} (1-h)^2+1$

$= \lim_{\substack{x\to 1-h\\ h\to}} (1-2h+h^2)+1 = 2$ Also, f(1) = 2

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

Hence, f(x) is continuous at x = 1 and at all points.

So, f(x) has no point of discontinuity.

11. $f(x) = \begin{cases} x^3 - 3, & if x \leq 2 \\ x^2 + 1, & if x > 2 \end{cases}$

SOLUTION

We observe that f(x) is continuous at all real numbers $x < 2$ and $x > 2$ as it is polynomial function. Now, continuity at x = 2 :

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2+1) = \lim_{\substack{x\to 2+h \\ h\to 0}} (2+h)^2 + 1$

$= \lim_{\substack{x\to 2+h \\ h\to 0}} (4+4h+h^2)+1 = 5$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3-3) = \lim_{\substack{x\to 2-h \\ h\to 0}} (2-h)^3 - 3$

$\lim_{\substack{x\to 2-h \\ h\to 0}} (2-12h+6h^2-h^3) - 3 = 5$

Also, $f(2) = 8-3 = 5 \therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

Hence, f(x) is continuous at x = 2 and at all points. So, f has no point of discontinuity.

12. $f(x) = \begin{cases} x^{10} - 1, & if x \leq 1 \\ x^2, & if x > 1 \end{cases}$

SOLUTION

Weobserve thatf(x)is continuousat real numbers x < 1 and x > 1 as it is polynomial function. Now, continuity at x = 1:

L.H.L.$= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = \lim_{\substack{x \to 1-h \\ h \to 0}} [(1-h)^{10} - 1] = 0$

R.H.L.$= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2) = \lim_{\substack{x\to 1+h \\ h\to 0}} (1+h)^2 = 1$

Also, $f(1) = 1^{10} - 1 = 0 \therefore$ L.H.L.$\neq$ R.H.L.$\neq f(1) \Rightarrow$ fis discontinuous at x = 1.

So, the only point of discontinuity of f (x) is 1.

13. Is the function defined by $f(x) = \begin{cases} x+5, & if x \le 1 \\ x-5, & if x > 1 \end{cases}$ a continuous function?

SOLUTION

We observe thatf(x) is continuous at all real numbers x < 1 and x > 1 as it is polynomial function. Now, continuity at x = 1:

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-5) = \lim_{\substack{x\to 1+h \\ h\to 0}} (1+h-5) = \lim_{\substack{x\to 1+h \\ h\to 0}} (h-4) = -4$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+5) = \lim_{\substack{x\to 1-h \\ h\to 0}} (1-h+5) = \lim_{\substack{x\to 1-h \\ h\to 0}} (6-h) = 6$

Thus, $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) \therefore$ f(x) is not continuous at x = 1.

14. Discuss the continuity of the function f, where f is defined by $f(x) = \begin{cases} 3, if & 0 \le x \le 1 \\ 4, if & 1 < x < 3 \\ 5, if & 3 \le x \le 10 \end{cases}$

SOLUTION

At x = 1 : L.H.L. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3 = 3$ and

R.H.L.$= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4 = 4 \therefore L.H.L. \neq R.H.L.$ at $x = 1$.

At x = 3 : L.H.L.$= \lim_{x\to 3^-} f(x) == \lim_{x\to 3^-} 4 = 4$ and

R.H.L.$== \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 5 = 5$

L.H.L.$\neq$ R.H.L. at x = 3.

Thus, function is not continuous at x = 1 and x = 3.

15. $f(x) = \begin{cases} 2x, if & x < 0 \\ 0, if & 0 \le x \le 1 \\ 4x, if & x > 1 \end{cases}$

SOLUTION

At x = 0 : L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2(0) = 0$

R.H.L.$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} 0 = 0$

Also, $f(0) = 0 \therefore$ L.H.L. = R.H.L = f (x)

So, f (x) is continuous at x = 0. At x = 1 :

L.H.L. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 0 = 0$ and $R.H.L. = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4(1) = 4$

Also, f (1) = 0 $\therefore$ L.H.L.$\neq$R.H.L. So, f(x) is discontinuous at x = 1.

16. . $f(x) = \begin{cases} -2, if & x \le -1 \\ 2x, if & -1 < x \le 1 \\ 2, if & x > 1 \end{cases}$

SOLUTION

At $x = -1$:

R.H.L.$= \lim_{x\to -1^+} f(x) = \lim_{\substack{x\to -1+h \\ h\to 0}} 2(-1+h) = -2$

L.H.L. $= \lim_{x\to -1^-} f(x) = \lim_{\substack{x\to -1-h \\ h\to 0}} (-2) = -2$

Also, $f(-1) = -2 \therefore \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$

Hence, $f(x)$ is continuous at x$=-1$. At x = 1:

R.H.L. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2) = 2$

L.H.L$= \lim_{x\to 1^-} f(x) = \lim_{\substack{x\to 1-h \\ h\to 0}} 2(1-h) = 2$

Also, f(1) = 2 $\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

Hence, f(x) is continuous at x = 1.

17. Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax+1, & if \quad x \le 3 \\ bx+3, & if \quad x > 3 \end{cases}$ is continuous at x= 3.

SOLUTION

At x= 3:

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = \lim_{\substack{x\to 3-h \\ h\to 0}} (a(3-h)+1)$

$= \lim_{x\to 3-h} (3a - ah + 1) = 3a + 1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} (bx+3) = \lim_{\substack{x\to 3+h \\ h\to 0}} (b(3+h)+3)$

$= \lim_{\substack{x\to 3+h \\ h\to 0}} (3b + bh + 3) = 3b + 3$ Also, $f(3) = 3a + 1$

Thus, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

f(x) is given as continuous at $x = 3] \Rightarrow 3b+3 = 3a+1 \Rightarrow 2 = 3(a-b) \Rightarrow a - b = \frac{2}{3}$

This is the required relation between a and b.

18. For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & if \quad x \le 0 \\ 4x+1, & if \quad x > 0 \end{cases}$ continuous at x= 0?

What about continuity at x = 1?

SOLUTION

Since f (x) is continuous at x = 0,

(i) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \lambda(x^2 - 2x) = \lambda(0-0) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 4x + 1 = 4(0) + 1 = 1$

As L.H.L. $\neq$ R.H.L. f(x) is continuous at x = 0 for no value of λ.

(ii) At x = 1 : $\lim_{x\to 1} f(x) = \lim_{x\to 1} 4x + 1 = 4\times 1 + 1 = 5$ and $f(1) = 4(1) + 1 = 5$ Thus, $\lim_{x\to 1} f(x) = f(1)$ for any value of λ.

Hence, f(x) is continuous at x = 1 for any real value of λ .

19. Show that the function defined by g(x) = x − [x] is discontinuous at all integral points. Here, [x] denotes the greatest integer less than or equal to x.

SOLUTION

Let $n \in I$. Then, $\lim_{x\to n^-}[x] = n-1$ [and $g(n) = n-n = 0$. [[n] = n because $n \in I$]

Now, $\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = \lim_{x\to n^-} x - \lim_{x\to n^-}[x] = n-(n-1) = 1$

and $\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x]) = \lim_{x\to n^+} x - \lim_{x\to n^+}[x] = n-n = 0$

Thus, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$.

Hence, g(x) is discontinuous at all integral points.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

SOLUTION

At $x = \pi$:

$\lim_{x\to\pi^+} f(x) = \lim_{\substack{x\to\pi+h\\h\to 0}} (\pi+h)^2 - \sin(\pi+h)+5$

$= \lim_{\substack{x\to\pi+\\h\to 0}} [(\pi^2+h^2+2\pi h)+\sinh+5] = \pi^2+5$

$\lim_{x\to\pi^-} f(x) = \lim_{\substack{x\to\pi-h\\h\to 0}} [(\pi-h)^2 - \sin(\pi-h)+5]$

$= \lim_{\substack{x\to\pi-h\\h\to 0}} (\pi^2+h^2-2\pi h) - \sin+5 = \pi^2+5$ Also, $f(\pi) = \pi^2+5$

Thus, R.H.L. = L.H.L. $= f(\pi)$. Function is continuous at x $=\pi$.

21. Discuss the continuity of the following functions :

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \cdot \cos x$

SOLUTION

(a) Let a be an arbitrary real number. Then, $f(a) = \sin a + \cos a$

$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\h\to 0}} [\sin(a+h)+\cos(a+h)]$

$= \lim_{\substack{x\to a+h\\h\to 0}} \{(\sin a\cosh + \cos a\sinh) + (\cos a\cosh - \sin a\sinh)\}$

$= \sin a(1) + \cos a(0) + \cos a(1) - \sin a(0) = \sin a + \cos a$

$\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\h\to 0}} [\sin(a-h)+\cos(a-h)]$

$= \lim_{\substack{x\to a-h\\h\to 0}} [(\sin a\cosh - \cos a\sinh) + (\cos a\cosh + \sin a\sinh)] = \sin a(1) - \cos a(0) + \cos a(1) + \sin a(0) = \sin a + \cos a.$

$\therefore \lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x) \Rightarrow f(x)$ is continuous at $x = a$. $\therefore f(x) =$ sin x + cos x is everywhere continuous.

(b) Let a be an arbitrary real number. Then $f(a) = \sin a - \cos a$

$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\h\to 0}} \sin(a+h) - \cos(a+h)$

$= \lim_{\substack{x\to a+h\\h\to 0}} \{(\sin a\cosh + \cos a\sinh) - (\cos a\cosh - \sin a\sinh)\}\pi$

$= \sin a(1) + \cos a(0) - \cos a(1) + \sin a(0) = \sin a - \cos a$

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a-h\\ h\to 0}} [(\sin(a-h) - \cos(a-h)]$

$= \lim\limits_{\substack{x\to a-h\\ h\to 0}} [(\sin a \cosh - \cos a \sinh) - (\cos a \cosh + \sin a \sinh)]$

$= \sin a(1) - \cos a(0) - \cos a(1) - \sin a(0) = \sin a - \cos a.$

$\therefore \lim\limits_{x\to a^-} f(x) = f(a) = \lim\limits_{x\to a^=} f(x)$

$\Rightarrow$ f(x) is continuous at $x = a$. $\therefore f(x) = \sin x - \cos x$ is everywhere continuous.

(c) Let a be an arbitrary real number. Then, $f(a) = \sin a \text{cosa}$

$\lim\limits_{x\to a^+} f(x) = \lim\limits_{\substack{x\to a+h\\ h\to 0}} [\sin(a+h)\cos(a+h)]$

$= \lim\limits_{\substack{x\to a+h\\ h\to 0}} [(\sin a \cosh + \cos a \sinh)(\cos a \cosh - \sin a \sinh)]$

$= ((\sin a(1) + \cos a(0)((\cos a)(1) - \sin a(0)) = \sin a \cos a.$

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a-h\\ h\to 0}} [\sin(a-h)\cos(a-h)]$

$= \lim\limits_{\substack{x\to a-h\\ h\to 0}} (\sin a \cosh - \cos a \sinh)(\cos a \cosh + \sin a \sinh)$

$= (\sin(a)(1) - \cos a(0))(\cos a(1) + \sin a(0))) = \sin a \cos a$

$\therefore \lim\limits_{x\to a^-} f(x) = f(a) = \lim\limits_{x\to a^+} f(x).$

$\Rightarrow f(x)$ is continuous at $x = a$. So, f(x)= $\sin x \cdot \cos x$ is everywhere continuous.

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

SOLUTION

(a) f(x) = cos x. Clearly, domain of f = R

Let a be an arbitrary real number, then f(a) = cos a.

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a-h\\ h\to 0}} \cos(a-h) = \lim\limits_{h\to 0}(\cos a \cosh + \sin a \sinh) = \cos a$

$\lim\limits_{x\to a^+} f(x) = \lim\limits_{\substack{x\to a+h\\ h\to 0}} \cos(a+h) = \lim\limits_{h\to 0}(\cos a \cosh - \sin a \sinh) = \cos a \therefore \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a) \Rightarrow f(x) = \cos x$ is continuous at a for all $a \in R$.

(b) $f(x) = \cos ecx \Rightarrow f(x) = \dfrac{1}{\sin x}$ and domain of $f = R - \{n\pi\}, n \in I$.

Also, $f(a) = \dfrac{1}{\sin a}$

$\lim\limits_{x\to a^+} \dfrac{1}{\sin x} = \dfrac{1}{\lim\limits_{\substack{x\to a+h\\ x\to 0}} \sin(a+h)}$

$= \lim\limits_{\substack{x\to a+h\\ x\to 0}} \dfrac{1}{\sin a \cosh + \cos a \sinh} = \dfrac{1}{\sin a \cos 0 + \cos a \sin(0)}$

$= \dfrac{1}{\sin a(1) + \cos a(0)} = \dfrac{1}{\sin a + 0} = \dfrac{1}{\sin a}$

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a-h\\ x\to 0}} \dfrac{1}{\sin(a-h)}$

$= \lim\limits_{\substack{x \to a-h \\ x \to 0}} \frac{1}{\sin a \cosh - \cos a \sinh} = \frac{1}{\sin a}$

$\therefore \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

Thus, cosec x is continuous at a for all $a \in R - \{n\pi\}, n \in I$.

(c) $f(x) = \sec x \Rightarrow f(x) = \frac{1}{\cos x}$ Clearly, domain of $f = R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

Also, $f(a) = \frac{1}{\cos a}$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{\substack{x \to a+h \\ h \to 0}} \frac{1}{\cos(a+h)}$

$= \lim\limits_{\substack{x \to a+h \\ h \to 0}} \frac{1}{\cos a \cosh - \sin a \sinh}$

$= \frac{1}{\cos a \cos 0 - \sin a \sin 0} = \frac{1}{\cos a(1) - \sin a(0)} = \frac{1}{\cos a}$

$\lim\limits_{x \to a^-} f(x) = \lim\limits_{\substack{x \to a-h \\ h \to 0}} \frac{1}{\cos(a-h)}$

$= \lim\limits_{\substack{x \to a-h \\ h \to 0}} \frac{1}{\cos a \cosh + \sin a \sinh} = \frac{1}{\cos a} \therefore \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$ '

Thus, sec x is continuous at a for all $a \in R - \{(2n+1\}\frac{\pi}{2}, n \in I$

(d) $f(x) = \cot x \; f(x) = \frac{1}{\tan x}$ and domain of $f = R - \{n\pi\}, n \in I$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{\substack{x \to a+h \\ h \to 0}} \frac{1}{\tan(a+h)}$

$= \lim\limits_{\substack{x \to a+h \\ h \to 0}} \frac{1}{\frac{\tan a + \tanh}{1 - \tan a \tanh}} = \frac{1}{\frac{\tan a + 0}{1 - \tan a \tan 0}} = \frac{1}{\frac{\tan a}{1-0}} = \frac{1}{\tan a}$

$\lim\limits_{x \to a^-} f(x) = \lim\limits_{\substack{x \to a-h \\ h \to 0}} \frac{1}{\tan(a-h)} = \lim\limits_{\substack{x \to a-h \\ h \to 0}} \frac{1}{\frac{\tan a - \tanh}{1 + \tan a \tanh}} = \frac{1}{\tan a}$

$\therefore \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

Thus, cot x is continuous at a for all $a \in R - n\pi, n \in I$.

23. Find all points of discontinuity of f, where

$f(x) = \begin{cases} \frac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

SOLUTION

At $x = 0, f(0) = 1$

L.H.L.$= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{\substack{x \to 0-h \\ h \to 0}} \frac{\sin(-h)}{-h} = 1$

R.H.L. $\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h \\ h\to 0}} (h+1) = 0+1 = 1$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ Thus, f(x) is continuous at x = 0.

When $x < 0$, sin x and x both are continuous. $\therefore \frac{\sin x}{x}$ is also continuous.

When $x > 0$, f(x) = x + 1 is a polynomial.

$\therefore$ f(x) is continuous. So, f(x) is not discontinuous at any point.

24. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ a continuous function?

SOLUTION

We have, $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h \\ h\to 0}} (0-h)^2 \sin\frac{1}{(0-h)} = \lim_{\substack{x\to 0-h \\ h\to 0}} \left(-h^2 \sin\frac{1}{h}\right)$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h \\ h\to 0}} (0+h)^2 \frac{1}{(0+h)} = \lim_{\substack{x\to 0+h \\ h\to 0}} h^2 \sin\frac{1}{h} = 0$$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \Rightarrow$ f is continuous at x = 0.

For x $\neq$0, f(x) is a continuous at every point. So, f(x) is a continuous function.

25. Examine the continuity off where f is defined by $f(x) = \begin{cases} \sin x - \cos x, & if \quad x \neq 0 \\ -1, & if \quad x = 0 \end{cases}$

SOLUTION

We have

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h \\ h\to 0}} [\sin(0-h) - \cos(0-h)]$$

$$= \lim_{\substack{x\to 0-h \\ h\to 0}} (-\sinh - \cosh) = -(0) - 1 = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0-h \\ h\to 0}} [\sin(0+h) - \cos(0+h)] = \lim_{h\to 0} (\sinh - \cosh)$$

lim f(x) = $\lim_{x\to a^+} [sin(0+h) - cos(0+h)] =$

lim(sin A-cos) $= 0 - 1 = -1$

Also,f(0) $= -1 \therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Hence, f(x) is continuous at x = 0. At $x < 0$, f(x) = sin x−cosx is continuous

At $x > 0$, f(x) = sin x−cosx is also continuous $\therefore f(x)$ is continuous at all $x \in R$.

Find the values of k so that the function is continuous at the indicated point in questions 26 to 29.

26. $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & if \quad x \neq \frac{\pi}{2} \\ 3, & if \quad x = \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$.

SOLUTION

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{\substack{x\to \frac{\pi}{2}-h \\ h\to 0}} \frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi - 2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} \frac{k\sin h}{\pi-\pi+2h}$$

$$= \lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} \frac{k\sin h}{2h} = \frac{k}{2}\lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} \frac{\sin h}{h} = \frac{k}{2}$$

$$\lim_{x\to\frac{\pi^+}{2}} f(x) = \lim_{\substack{x\to\frac{\pi}{2}+h \\ h\to 0}} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{\substack{x\to\frac{\pi}{2}+h \\ h\to 0}} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2}\lim_{\substack{x\to\frac{\pi}{2}+h \\ h\to 0}} \frac{\sin h}{h} = \frac{k}{2}$$

Also, $f\left(\frac{\pi}{2}\right) = 3$. For continuity at $x = \frac{\pi}{2}$, we have

$$\lim_{x\to\frac{\pi^-}{2}} f(x) = \lim_{x\to\frac{\pi^+}{2}} f(x) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$$

27. $f(x) = \begin{cases} kx^2 & if \quad x \le 2 \\ 3, & if \quad x > 2 \end{cases}$ at $x = 2$.

SOLUTION

We have, $f(2) = 4k$

$$\lim_{x\to 2^+} f(x) = \lim_{\substack{x\to 2+h \\ h\to 0}} 3 = 3$$

$\lim_{x\to 2^-} f(x) = \lim_{\substack{x\to 2-h \\ h\to 0}} (2-h)^2 k = 4k$ For continuity at x = 2, we have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$$\Rightarrow 4k = 3 \Rightarrow k = \frac{3}{4}$$

28. $f(x) = \begin{cases} kx+1, & if \quad x \le \pi \\ \cos, & if \quad x > \pi \end{cases}$ at $x = \pi$.

SOLUTION

$$\lim_{x\to\pi^+} f(x) = \lim_{\substack{x\to\pi+h \\ h\to 0}} f(\pi+h)$$

$$= \lim_{\substack{x\to\pi+h \\ h\to 0}} \cos(\pi+h) = \lim_{\substack{x\to\pi+h \\ h\to 0}} -\cos h = -\cos(0) = -1$$

$\lim_{x\to\pi^-} f(x) = \lim_{\substack{x\to\pi-h \\ h\to 0}} k(\pi-h)+1 = k\pi+1$ and $f(\pi) = k\pi+1$

Since the given function is continuous at $x = \pi$,

$$\therefore \lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^-} f(x) = f(\pi)$$

$$\Rightarrow k+1 = -1 \Rightarrow k\pi = -1-1 \Rightarrow k\pi = -2 \Rightarrow k = \frac{-2}{\pi}$$

29. $f(x) = \begin{cases} kx+1, & if \quad x \le 5 \\ 3x-5, & if \quad x > 5 \end{cases}$ at $x = 5$

SOLUTION

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (kx+1)$

$= \lim_{\substack{x\to 5-h \\ h\to 0}} (k(5-h)+1) = k(5-0)+1 = 5k+1$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} (3x-5) = \lim_{\substack{x\to 5+h \\ h\to 0}} (3(5+h)-5) = 3(5+0)-5 = 10$

For continuity at x= 5, $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5) \Rightarrow 5k+1 = 10 \Rightarrow 5k = 9 \Rightarrow k = \frac{9}{5}$

30. Find the values of a and b such that the function defined by $f(x) = \begin{cases} 5, & if \quad x \le 2 \\ ax+b, & if \quad 2 < x < 10 \\ 21, & if \quad x \ge 10 \end{cases}$ is a continuous function.

SOLUTION

Since f is continuous at all x, so f is continuous at x = 2, 10.

At x = 2 : $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (5) = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax+b)$

$= \lim_{\substack{x\to 2+h \\ h\to 0}} (a(2+h)+b) = a(2+0)+b = 2a+b$ and $f(2) = 5$

For continuity, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow 5 = 2a+b = 5 \Rightarrow 2a+b = 5$ .....(i)

At x = 10 : $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} (ax+b)$

$= \lim_{\substack{x\to 10+h \\ h\to 0}} (a(10-h)+b) = a(10-0)+b = 10a+b$

$\lim_{x\to 10^+} f(x) = \lim_{x\to 10^+} (21) = 21$ $f(10) = 21$

For continuity, $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$\Rightarrow 10a+b = 21 \Rightarrow 10a+b = 21$ ...(ii)

Subtracting (i) from (ii), we get 8a = 16 ⇒ a=2

Putting a = 2 in (i), we get 2(2) + A = 5 ⇒ b = 5− 4 = 1 Hence, a =2, A = 1.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

SOLUTION

Let f(x) = $\cos(x^2)$. Domain of f = R.

Let a be any arbitrary real number.

Then, $\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h \\ h\to 0}} \cos(a+h)^2 = \cos a^2$

Then, $\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h \\ h\to 0}} \cos(a-h)^2 = \cos a^2$ and $f(a) = \cos a^2$

Thus, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a) \forall a \in R$.

$\therefore f(x) = \cos(x^2)$ is continuous at$a \forall a \in R$.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

SOLUTION

We know that cosine fiinction is everywhere continuous and also modulus function is continuous. Therefore, $|\cos x|$ is everywhere continuous.

33. Examine that $sin|x|$ is a continuous function.

SOLUTION

Let $f(x) = |x|$ and $g(x) = \sin x$. Then, $(gof)(x) = g[f(x)] = g(|x|) = \sin|x|$

Now, f and g being continuous, it follows that their composite function (gof) is continuous.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

SOLUTION

We have,

$$f(x) = \begin{cases} -(x) - [-(x+1)], & if \quad x < -1 \\ -(x) - (x+1), & if \quad -1 \le x < 0 \\ (x) - (x+1), & if \quad x \ge 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, & if \quad x < -1 \\ -2x - 1, & if \quad -1 \le x < 0 \\ -1, & if \quad x \ge 0 \end{cases}$$

At $x = -1 :$ $\lim_{x \to -1^-} f(x) = 1$

$\lim_{x \to -1^+} f(x) = \lim_{\substack{x \to -1+h \\ h \to 0}} (-2(-1+h) - 1) = 1$ $f(-1) = -2(-1) - 1 = 1$

Thus, $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1) \Rightarrow$ f is continuous at $x = -1$

At $x = 0 :$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x - 1) = \lim_{\substack{x \to 0-h \\ h \to 0}} (-2(-h) - 1) = -1$

Also, $f(0) = -1$

Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \Rightarrow$ f is continuous at $x = 0$. Also, f being a constant is continuous when

$x < -1$ or when x > 0. $\therefore$ f is continuous for all $x \in R$ Hence, there is no point of discontinuity.

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