NCERT - Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x-1)^2 + 3$

(ii) $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$

(iv) $g(x) = x^3 + 1$

SOLUTION

(i) We have, $f(x) = (2x-1)^2 + 3$ for all $x \in R$.

Since $(2x-1)^2 \geq 0 \Rightarrow (2x-1)^2 + 3 \geq 3 \therefore$ Minimum $f(x) = 3$, which occurs when $2x - 1 = 0$ i.e., when $x = 1/2$

Value of $f(x)$ has no maximum value because $f(x) \to \infty$ as $|x| \to \infty$

(ii) We have $f(x) = 9x^2 + 12x + 2 = 9\left(x^2 + \frac{4}{3}x\right) + 2 = 9\left\{x^2 + \frac{4}{3}x + \frac{4}{9}\right\} + 2 - 4 = 9\left(x + \frac{2}{3}\right)^2 - 2$

Since $\left(x + \frac{2}{3}\right)^2 \geq 0 = 9\left(x + \frac{2}{3}\right)^2 - 2 \geq -2$

$\Rightarrow f\langle x) \geq -2$ for all $x \in R$.

$\therefore$ Minimum, $f(x) = -2$, which occurs when $x + \frac{2}{3} = 0$, i.e., when $x = \frac{-2}{3}$. $f(x)$ has no maximum value because $f(x) \to \infty$ as $|x| \to \infty$.

(iii) We have, $f(x) = 10 - (x-1)^2$ for aI $x \in R$. $(x-1)^2 \geq 0 \; \forall \, x \in R \Rightarrow -(x-1)^2 \leq 0 \; \forall \, x \in R$

$\Rightarrow 10 - (x-1)^2 \leq 10 \; \forall \, x \in R \therefore$ Maximum $f(x) = 10$, which occurs when $x - 1 = 0$ i.e., when $x = 1$. $f(x)$ has no minimum value because $f(x) \to -\infty$ as $|x| \to \infty$.

(iv) We have, $g(x) = x^3 + 1$ As $x \to \infty, g(x) \to \infty$ and as $x \to -\infty, g(x) \to -\infty$ $g(x)$ has neither a maximum nor a minimum value.

2. Find-the maximum and minimum values, if any, of the following functions given by (i) $f(x\rangle = |x+2| - 1$

(ii) $g(x) = -|x+1| + 3$

(iii) $h(x) = \sin(2x) + 5$

(iv) $f(x\rangle = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

SOLUTION

(i) We have, $f(x) = |x+2| - 1$ for all $x \in R$. Since, $|x+2| \geq 0 \Rightarrow |x+2| - 1 \geq -1$ Minimum $f(x) = -1$, which occurs when $x + 2 = 0$ i.e., when $x = -2$. $f(x)$ has no maximum value because $f(x\rangle \to \infty$ as $|x| \to \infty$.

(ii) We have, $g(x) = -|x+1| + 3$ for all $x \in R$.

Since, $|x+1| \geq 0 \Rightarrow -|x+1| \leq 0 \Rightarrow -|x+1| + 3 \leq 3 \therefore$ Maximum value of $g(x)$ is 3, which occurs when $x + 1 = 0$, i.e., when $x = -1$. $g(x)$ has no minimum value because $g(x) \to -\infty$ as $|x| \to \infty$.

(iii) We have, $h(x) = \sin 2x + 5 \; \forall \, x \in R$

We know that, $-1 \leq \sin 2x \leq 1$ for all $x \in R \Rightarrow 5 - 1 \leq 5 + \sin 2x \leq 5 + 1$ for all $x \in R \Rightarrow 4 \leq f(x) \leq 6$ for all $x \in R$.

$\therefore$ Maximum value of $f(x) = 6$, which occurs when $\sin 2x = 1$ and minimum value of $f(x) = 4$, which occurs when $\sin 2x = -1$.

(iv) We have, $f(x) = |\sin 4x + 3| \forall x \in R$. We know that, $-1 \leq \sin 4x \leq 1$ for all $x \in R$

$\Rightarrow 3 - 1 \leq \sin 4x + 3 \leq 1 + 3$ for all $x \in R$. $\Rightarrow |2| \leq |\sin 4x + 3| \leq |4| \forall \, x \in R$

$\Rightarrow 2 \leq f(x) \leq 4 \; \forall \, x \in R \therefore$ Minimum value of $f(x) = 2$, which occurs when $\sin 4x = -1$ and maximum value of $f(x) = 4$, which occurs when $\sin 4x = 1$.

(v) We have, $h(x) = x+1, \forall -1 < x < 1$. $-1 < x < 1 = -1+1 < x+1 < 1+1 \Rightarrow 0 < x+1 < 2$ Here, range of $h = (0,\ 2)$

There is no definite value for maximum or minimum of $h(x)$. $\therefore$ h has neither a maximum nor a minimum value.

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$

(ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \dfrac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x, 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

(vi) $g(x) = \dfrac{x}{2} + \dfrac{2}{x}, x > 0$

(vii) $g(x) = \dfrac{1}{x^2+2}$

(viii) $f(x) = x\sqrt{1-x}, x > 0$

SOLUTION

(i) We have, $f(x) = x^2 \Rightarrow f'(x) = 2x$ For critical points, $f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

The point at which extremum may occur is $x = 0$ Now, $f''(x) = 2 \Rightarrow f''(0) = 2 > 0$ $\therefore$ f has a local minima at $x = 0$ and local minimum value is $f(0) = 0^2 = 0$.

(ii) We have, $g(x) = x^3 - 3x \Rightarrow g'(x) = 3x^2 - 3$

For critical points, $g(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$ The points at which extremum may occur are -1 and 1 . $g''(x) = 6x$

$\Rightarrow g''(-1) = 6(-1) = -6 < 0$ $\therefore$ g has a local maxima at $x = -1$ and local maximum value at $x = -1$ is $g(-1) = (-1)^3 - 3(-1) = -1+3 = 2$.

$g''(1) = 6 \times 1 = 6 > 0$ $\therefore$ g has a local minima at $x = 1$ and local minimum value at $x = 1$ is $g(1) = 1^3 - 3 \times 1 = -2$.

(iii) We have, $h(x) = \sin x + \cos x, 0 < x < \dfrac{\pi}{2} \Rightarrow h'(x) = \cos x - \sin x$ for all $x \in \left(0,\ \dfrac{\pi}{2}\right)$

For critical points, $h'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$

The point at which extremum may occur is $x = \dfrac{\pi}{4}$ $h''(x) = -\sin x - \cos x$

$$\Rightarrow h''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{-2}{\sqrt{2}} < 0$$

$\therefore$ h has a local maxima at $x = \dfrac{\pi}{4}$ and local maximum value at $x = \dfrac{\pi}{4}$ is $h\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

(iv) We have, $f(x) = \sin x - \cos x,\ 0 < x < 2\pi \Rightarrow f'(x) = \cos x + \sin x$

For critical points, $f'(x) = 0 \Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1$

$$\Rightarrow x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4} \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$\therefore$ The points at which extremum may occur are x $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$ $f''(x) = -\sin x + \cos x$

$$\Rightarrow f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0$$

$\therefore f$ has local maxima at $x = \dfrac{3\pi}{4}$ and local maximum value at $x = \dfrac{3\pi}{4}$ is $f\left(\dfrac{3\pi}{4}\right) = \sin\dfrac{3\pi}{4} - \cos\dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}} - \left(-\dfrac{1}{\sqrt{2}}\right) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

Further $f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$= -\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

$\therefore f$ has local minima at $x = \frac{7\pi}{4}$ and local minimum value at $x = \frac{7\pi}{4}$ is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

(v) We have, $f(x) = x^3 - 6x^2 + 9x + 15, x \in R \Rightarrow f'(x) = 3x^2 - 12x + 9, x \in R$ For critical points, $f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow 3(x-1)(x-3) = 0 \Rightarrow x = 1, x = 3$

$\therefore$ The points where extremum may occur are $x = 1$ and $x = 3$. $f''(x) = 6x - 12$, $x \in R$. $\Rightarrow f''(1) = 6 \times 1 - 12 = -6 < 0 \therefore f$ has a local maxima at $x = 1$ and local maximum value at $x = 1$ is $f(1) = 1 - 6 + 9 + 15 = 19$. $f''(3) = 6 \times 3 - 12 = 6 > 0$

$\therefore f$ has a local minima at $x = 3$ and local minimum value at $x = 3$ is $f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 + 15 = 15$.

(vi) Given, $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0 \Rightarrow g'(x) = \frac{1}{2} + \left(-\frac{2}{x^2}\right), x > 0$

For critical points, $g'(x) = 0 \Rightarrow \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = -2, 2$

$\therefore$ The only point where extremum may occur is $x = 2$. $g''(x) = -2(-2)x^{-3}, x > 0$ and $g''(2) = 4(2)^{-3} = \frac{4}{8} = \frac{1}{2} > 0 \therefore f$ has a local minima at $x = 2$ and local minimum value is $g(2) = \frac{2}{2} + \frac{2}{2} = 2$

(vii) Given, $\Rightarrow g'(x) = \frac{-2x}{(x^2+2)^2}$ $(x^2+2)^2$For critical points, $g'(x) = 0$

$\Rightarrow \frac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$ $g''(x) = \frac{6x^2-4}{(x^2+2)^3} \Rightarrow g''(0) = \frac{-4}{8} < 0$

$\therefore g$ has a local maxima at $x = 0$ and local maximum value is $g(0) = \frac{1}{0+2} = \frac{1}{2}$

(viii) We have, $f(x) = x\sqrt{1-x}, 0 < x \le 1$

$\Rightarrow f'(x) = \frac{x(-1)}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$

For critical points, $f'(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow x = \frac{2}{3}$.

$\therefore$ The point at which extremum may occur is $x = 2/3$. $f''(x) = \frac{\frac{1}{2}\left\{(\sqrt{1-x})(-3) - \frac{(2-3x)(-1)}{2\sqrt{1-x}}\right\}}{(1-x)}$

$= \frac{1}{2}\left[\frac{2(1-x)(-3)+2-3x}{2\sqrt{1-x}(1-x)}\right]$

$\Rightarrow f''\left(\frac{2}{3}\right) = \frac{1}{2}\left[\frac{2\left(1-\frac{2}{3}\right)(-3)+2-3\times\left(\frac{2}{3}\right)}{2\sqrt{1-\frac{2}{3}}\left(1-\frac{2}{3}\right)}\right]$

$= \frac{1}{2}\left(\frac{\frac{-2\times1\times3}{3}+2-2}{2\sqrt{\frac{1}{3}}\left(\frac{1}{3}\right)}\right) < 0$

$\therefore f$ has local maxima at $x=\frac{2}{3}$ and local maximum value is $f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}$

4. Prove that the following functions do not have maxima or minima:
   (i) $f(x)=e^x$
   (ii) $g(x)=\log x$
   (iii) $h(x)=x^3+x^2+x+1$

   SOLUTION

   (i) We have, $f(x\rangle = e^x \Rightarrow f'(x)=e^x \forall x\in R$ $f'(x)=e^x>0 \forall x\in R \Rightarrow f$ has no critical point.
   Thus, there is no point at which $f$may have an extremum. $\therefore f$ has neither a maximum nor a minimum value.

   (ii) We have, $g(x)=\log x,\ x>0 \Rightarrow\ g'(x)=\frac{1}{x}, x>0$ $g'(x)=\frac{1}{x}\neq 0$ for all $x\in(0,\ \infty)\Rightarrow g$ has no critical point.
   Thus, there is no point at which g may have an extremum. $\therefore g$ has neither a maximum nor a minimum value.

   (iii) We have $h(x)=x^3+x^2+x+1, x\in R \Rightarrow h'(x)=3x^2+2x+1, x\in R$
   For Critical points, $h'(x)=0$ $3x^2+2x+1=0\Rightarrow x=\frac{-2\pm\sqrt{-8}}{6}$ which is non-real $\Rightarrow h$ has no critical point.
   Thus, there is no point at which h may have an extremum. $\therefore h$ has neither a maximum nor a minimum value.

5. Find the absolute maximum value and the absolute minimum value of thye following functions in the given intervals:
   (i) $f(x)=x^3, x\in[-2,2]$

   (ii) $f(x)=\sin x+\cos x, x\in[0,\pi]$

   (iii) $f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right]$

   (iv) $f(x)=(x-1)^2+3, x\in[-3,1]$

   SOLUTION

   (i) $f(x)=x^3, x\in[-2,2]\& \Rightarrow\ f'(x)=3x^2$ For critical points, $f'(x)=0\Rightarrow 3x^2=0\Rightarrow x=0\in[-2,2]$
   Hence, for finding the absolute maximum value and the absolute minimum value, we have to evaluate $f(0), f(-2)$ and $f(2)$. $f(0)=0^3=0,\ f(-2)=(-2)^3=-8$ and $f(2)=2^3=8$
   $\therefore$ Absolute maximum value of $f(x)$ is 8 at $x=2$and absolute minimum value of $f(x)$is -8 at $x=-2$.

   (ii) We have, $f(x)=\sin x+\cos x,\ x\in[0,\pi]\Rightarrow\ f'(x)=\cos x-\sin x$ For critical points, $f'(x)=0\Rightarrow\cos x-\sin x=0\Rightarrow\tan x=1\Rightarrow x=\frac{\pi}{4}\in[0,\pi]$
   Hence, for finding the absolute maximum value and the absolute minimum value, we have to evaluate $f(0), f(\pi)$ and $f\left(\frac{\pi}{4}\right)$.
   $f(0)=\sin 0+\cos 0=0+1=1,$
   $f(\pi)=\sin\pi+\cos\pi=0-1=-1$ $f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

   $\therefore$ Absolute maximum value of $f(x)$ is $\sqrt{2}$ at $x=\frac{\pi}{4}$and absolute minimum value of $f(x)$is -1 at $x=\pi$

   (iii) We have , $f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right]\Rightarrow f'(x)=4-x$ For critical points, $f'(x)=0\& \Rightarrow 4-x=0\Rightarrow x=4\in\left[-2,\frac{9}{2}\right]$
   Hence, for finding the absolute maximum value and the absolute minimum value, we have to evaluate $f(-2), f\left(\frac{9}{2}\right)$ and $f(4)$.
   $f(-2)=4(-2)-\frac{1}{2}(-2)^2=-8-2=-10,$

$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=\frac{63}{8}$ $f(4)=4\times4-\frac{1}{2}(4)^2=16-8=8$

$\therefore$ Absolute maximum value of $f(x)$ is 8 at $x=4$and absolute minimum value of $f(x)$is -10 at $x=-2$

(iv) We have , $f(x)=(x-1)^2+3, x\in[-3,1]\Rightarrow f'(x)=2(x-1)$ For critical points, $f'(x)=0\Rightarrow 2(x-1)=0\Rightarrow x=1\in[-3,1]$
Hence, for finding the absolute maximum value and the absolute minimum value of $f(x)$, we have to evaluate $f(-3)$ and $f(1)$.
$f(-3)=(-3-1)^2+3=19$and $f(1)=(1-1)^2+3=3$
$\therefore$ Absolute maximum value of $f(x)$ is 19 at $x=-3$and absolute minimum value of $f(x)$is 3 at $x=1$

6. Find the maximum profit that a company can make , if the profit function is given by $p(x) = 41 - 72x - 18x^2$.

SOLUTION

We have, $p(x) = 41 - 72x - 18x^2 \Rightarrow p'(x) = -72 - 36x$

Now, for critical points, $p'(x) = 0 \Rightarrow -72 - 36x = 0 \Rightarrow x = -2$ $p''(x) = -36 < 0 \Rightarrow p''(-2) = -36 < 0$

$\therefore$ Profit is maximum at $x=-2$ and maximum profit is $p(-2)=41-72(-2)-18(-2)^2=41+144-72=185-72=113$ units

7. Find both the maximum value and the minimum value of $3x^4-8x^3+12x^2-48x+25$ on the interval $[0,3]$.

SOLUTION

Let $f(x)=3x^4-8x^3+12x^2-48x+25,\ x\in[0,3]$

$\Rightarrow f'(x)=12x^3-24x^2+24x-48=12\left\{x^3-2x^2+2x-4\right\}=12\left\{x^2(x-2)+2(x-2)\right\}=12(x-2)\left(x^2+2\right)$

For critical points, $f'(x)=0\Rightarrow 12(x-2)\left(x^2+2\right)=0\Rightarrow x-2=0\Rightarrow x=2\in[0,3]$

So, to find the maximum and minimum values, we have to evaluate $f(0),f(3)$ and $f(2)$.

Now, $f(0)=25$ $f(2)=3\times2^4-8\times2^3+12\times2^2-48\times2+25=-39$ $f(3)=3\times3^4-8\times3^3+12\times3^2-48\times3+25=16$

$\therefore$ Maximum value of $f(x)$ is 25 at $x=0$ and minimum value of $f(x)$ is -39 at $x=2$.

8. At what points in the interval $[0,2\pi]$, does the function $\sin 2x$ attain its maximum value?

SOLUTION

Let $f(x)=\sin 2x,\ 0\le x\le 2\pi\Rightarrow f'(x)=2\cos 2x$

For critical points, $f'(x)=0\Rightarrow 2\cos 2x=0\Rightarrow \cos 2x=0\Rightarrow 2x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}\Rightarrow x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

So, for finding the maximum and minimum values, we have to evaluate $f(0),f(2\pi),f\left(\frac{\pi}{4}\right),f\left(\frac{3\pi}{4}\right),f\left(\frac{5\pi}{4}\right),f\left(\frac{7\pi}{4}\right)$

Now, $f(0)=\sin(2\times0)=0, f(2\pi)=\sin(2\times2\pi)=0,\ f\left(\frac{\pi}{4}\right)=\sin\left(2\times\frac{\pi}{4}\right)=\sin\frac{\pi}{2}=1$

$f\left(\frac{3\pi}{4}\right)=\sin\left(2\times\frac{3\pi}{4}\right)=\sin\frac{3\pi}{2}=\sin\left(\pi+\frac{\pi}{2}\right),=-\sin\frac{\pi}{2}=-1$

$f\left(\frac{5\pi}{4}\right)=\sin\left(2\times\frac{5\pi}{4}\right)=\sin\frac{5\pi}{2}=\sin\frac{\pi}{2}=1$ and $f\left(\frac{7\pi}{4}\right)=\sin\left(2\times\frac{7\pi}{4}\right)=\sin\frac{7\pi}{2}=\sin\frac{-\pi}{2}=-1$

$\therefore$ Maximum value of $f(x)$ is 1 at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

9. What is the maximum value of the function $\sin x+\cos x$?

SOLUTION

Let $f(x)=\sin x+\cos x, x\in R\Rightarrow f'(x)=\cos x-\sin x$

For critical points, $f(x)=0\Rightarrow \cos x-\sin x=0\Rightarrow \tan x=1\Rightarrow x=\frac{\pi}{4},\frac{5\pi}{4}$

For maximum value, we have to evaluate $f\left(\frac{\pi}{4}\right)$ and

$f\left(\frac{5\pi}{4}\right)$ $f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\sqrt{2}$

$f\left(\frac{5\pi}{4}\right)=\sin\frac{5\pi}{4}+\cos\frac{5\pi}{4}=\sin\left(\pi+\frac{\pi}{4}\right)+\cos\left(\pi+\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=-\sqrt{2}$

$\therefore$ Maximum value of the function is $\sqrt{2}$ at $x=\frac{\pi}{4}$

10. Find the maximum value of $2x^3-24x+107$ in the interval $[1,3]$. Find the maximum value of the same function in $[-3,\ -1]$.

SOLUTION

Let $f(x)=2x^3-24x+107, 1\le x\le3\Rightarrow f'(x)=6x^2-24$

For critical points, $f'(x)=0\Rightarrow 6x^2-24=0\Rightarrow 6(x^2-4)=0\Rightarrow x^2=4\Rightarrow x=\pm2\Rightarrow x=2\in[1,3]$

So, for maximum or minimum value, we have to evaluate $f(1)$,$f(2)$ and $f(3)$ Now, $f(1)=2\times1^3-24\times1+107=85$, $f(2)=2\times2^3-24\times2+107=75$, $f(3)=2\times3^3-24\times3+107=89$ $\therefore$ Maximum value of $f(x)$ is 89 at $x=3$

If we consider the function in the interval $[-3,\ -1]$, then we take $x=-2 \in [-3,\ -1]$. So, we evaluate $f(-1), f(-2)$, and $f(-3)$.
$f(-1)=2(-1)^3-24(-1)+107=129$ $f(-2)=2(-2)^3-24(-2)+107=139$

$f(-3)=2(-3)^3-24(-3)+107=125$

$\therefore$ Maximum value of $f(x)$ is 139 at $x=-2$.

11. It is given that at $x=1$, the function x^4-63x^2+ax+9 attains its maximum value, on the interval $[0,2]$. Find the value of a.

SOLUTION

$f(x)=x^4-62x^2+ax+9, 0 \le x \le 2 \Rightarrow f'(x)=4x^3-124x+a$

As $f(x)$ attains maximum value at $x=1 \in [0,2]$, $\therefore$ We must have $f'(1)=0 \Rightarrow 4-124+a=0 \Rightarrow a=120$

Thus, we have $f(x)=x^4-62x^2+120x+9$ $f(0)=9, f(1)=1-62+120+9=68$ and $f(2)=2^4-62\times 2^2+120\times 2+9=17$
Clearly, $f(1)$ is maximum. Hence, $a=120$.

12. Find the maximum and minimum values of $x+\sin 2x$ on $[0,2\pi]$.

SOLUTION

Let $f(x)=x+\sin 2x, 0 \le x \le 2\pi \Rightarrow f'(x)=1+2\cos 2x, 0<x<2\pi$.

For critical points, $f'(x)=0 \Rightarrow 1+2\cos 2x=0 \Rightarrow \cos 2x=-\frac{1}{2} \Rightarrow \cos 2x=-\cos\frac{\pi}{3}$ [If $0<x<2\pi$, then $0<2x<4\pi$]

$\Rightarrow \cos 2x=\cos\left(\pi-\frac{\pi}{3}\right), \cos\left(\pi+\frac{\pi}{3}\right), \cos\left(3\pi-\frac{\pi}{3}\right)$, $\cos\left(3\pi+\frac{\pi}{3}\right) \Rightarrow 2x=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\frac{10\pi}{3} \Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$

So, for finding maximum and minimum values, we evaluate $f(x)$ at $0, 2\pi, \frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$.

Now, $f(0)=0+\sin 0=0$ $f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$ $f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\sin\left(\pi-\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}+\sin\left(\pi+\frac{\pi}{3}\right)=\frac{2\pi}{3}-\sin\frac{\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\sin\left(2\pi+\frac{2\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{2\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$ and $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}+\sin\left(3\pi+\frac{\pi}{3}\right)$

$=\frac{5\pi}{3}-\sin\frac{\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$

Thus, maximum value of $f(x)$ is 2π at $x=2\pi$ and minimum value of $f(x)$ is 0 at $x=0$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

SOLUTION

Let the two numbers be x and $24-x$ Let $p=x(24-x) \Rightarrow p=24x-x^2$

$\Rightarrow \frac{dp}{dx}=24-2x$

For p to be largest $\frac{dp}{dx}=0 \Rightarrow 24-2x=0 \Rightarrow x=12$ and $\frac{d^2p}{dx^2}=-2, \left(\frac{d^2p}{dx^2}\right)_{x=12}=-2<0$

$\Rightarrow p$ has maximum value at $x=12$. So, the required parts are 12and$24-12$ i.e., 12and12.

14. Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

SOLUTION

We have two numbers x and y such that $x+y=60$ (i)

Let $P=xy^3 \Rightarrow P=(60-y)y^3$ [from(i)] $\Rightarrow P=60y^3-y^4 \Rightarrow \frac{dP}{dy}=180y^2-4y^3$

For maximum P, we must have $\frac{dP}{dy}=0$

$\Rightarrow 180y^2-4y^3=0 \Rightarrow 4y^2(45-y)=0 \Rightarrow y=45$ ($y=0$ is not possible for xy^3 to be maximized)

Also, $\frac{d^2P}{dy^2} = 360y - 12y^2$ and $\left(\frac{d^2P}{dy^2}\right)_{y=45} = 360 \times 45 - 12 \times (45)^2 < 0$

Therefore, P is maximum when $y = 45$. $\therefore$ Required numbers are $x = 60 - y = 60 - 45 = 15$ and $y = 45$.

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.

SOLUTION

We have numbers x and y and let $P = x^2y^5$ and $x + y = 35$ (i) $P = (35-y)^2y^5$ [from (i)]

Now, $\frac{dP}{dy} = (35-y)^2(5y^4) + y^5 \times 2(35-y)(-1) = y^4(35-y)\{5(35-y) - 2y\} = y^4(35-y)(175-7y)$

For maximum P, $\frac{dP}{dy} = 0 \Rightarrow y^4(35-y)(175-7y) = 0 \Rightarrow 175 - 7y = 0 \Rightarrow y = 25$

Now, $\frac{d^2P}{dy^2} = 4(35-y)(175-7y)y^3 + y^4(-1)(175-7y) + y^4(35-y)(-7) \Rightarrow \left(\frac{d^2P}{dy^2}\right)_{y=25} < 0$

So, P has a maximum value at $y = 25$. $\therefore$ Required numbers are $x = 35 - 25 = 10$ and $y = 25$.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

SOLUTION

Let the numbers be x and $16-x$ and let $S = x^3 + (16-x)^3 \Rightarrow S = x^3 + (16-x)^3 \Rightarrow \frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1)$

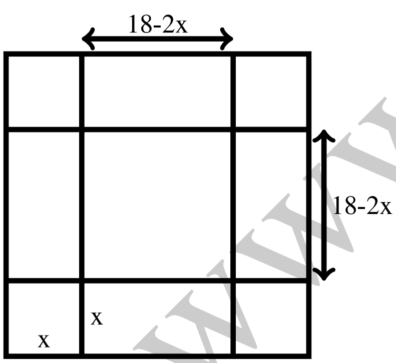
For minimum S, $\frac{dS}{dx} = 0 \Rightarrow 3x^2 - 3(16-x)^2 = 0 \Rightarrow x^2 - (256 + x^2 - 32x) = 0 \Rightarrow 32x = 256 \Rightarrow x = 8$

$\frac{d^2S}{dx^2} = 6x + 6(16-x)$ and $\left(\frac{d^2S}{dx^2}\right)_{x=8} = 96 > 0$ $\therefore$ S has a minimum value at $x = 8$. Hence, the required numbers are 8and8 .

17. A square piece of tin of side 18cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

SOLUTION

Let $x$cm be the length of each side of the square which is to be cut off from each corner of the square tin sheet of side 18cm.



Let V be the volume of the open box formed by folding up the flaps, then $V = x(18-2x)(18-2x\rangle = 4x(9-x)^2 = 4(x^3 - 18x^2 + 81x)$ (i)

Differentiating (i) w.r.t. x, we get $\frac{dV}{dx} = 4(3x^2 - 36x + 81) = 12(x^2 - 12x + 27)$ For maximum/minimum volume, $\frac{dV}{dx} = 0 \Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow 12(x-3)(x-9) = 0 \Rightarrow x = 3, 9$ but $0 < x < 9 \Rightarrow x = 3$

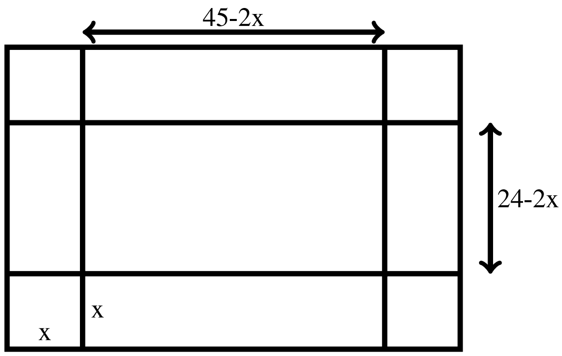
$\frac{d^2V}{dx^2} = 12(2x - 12) = 24(x-6)$ and $\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3-6) = -72 < 0 \Rightarrow V$ has a maximum value at $x = 3$.

Hence, the volume of the box is maximum when the side of the square to be cut off is 3cm.

18. A rectangular sheet of tin 45cmby24cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

SOLUTION

Let the side of the square to be cut from each of the four corners be $x$cm, then the base of the box has dimensions $(45-2x)$ cm and $(24-2x)$ cm and height of the box is x cm.



Let V be the corresponding volume of the box, then $V = x(24-2x)(45-2x) = x(4x^2-138x+1080) = 4x^3-138x^2+1080x$

$\Rightarrow \dfrac{dV}{dx} = 12x^2-276x+1080$

For maximum/minimum volume, $\dfrac{dV}{dx} = 0 \Rightarrow 12x^2-276x+1080 = 0$

$\Rightarrow x^2-23x+90 = 0 \Rightarrow (x-18)(x-5) = 0$

$\Rightarrow x = 5, x \neq 18$ $\dfrac{d^2V}{dx^2} = 24x-276$ and $\left(\dfrac{d^2V}{dx^2}\right)_{x=5} = 24\times 5-276 < 0$

$\therefore V$ has maximum value at $x = 5$.

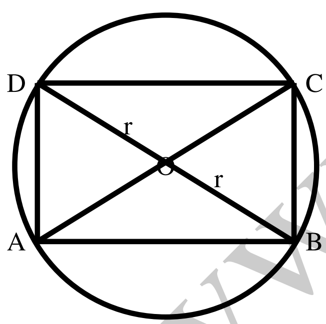
So, a square of side 5cm should be cut from each corner for the box to have a maximum volume.

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

SOLUTION

Let $ABCD$ be a rectangle inscribed in the given circle of radius r having centre at O.

Let one side of the rectangle be x, then the other side $= \sqrt{(2r)^2-x^2} = \sqrt{4r^2-x^2}$ (; an angle in the semicircle) Let A be the corresponding area of the rectangle, then



$A = x\sqrt{4r^2-x^2}, 0 < x < 2r \Rightarrow \dfrac{dA}{dx} = \dfrac{x(-2x)}{2\sqrt{4r^2-x^2}} + \sqrt{4r^2-x^2} = \dfrac{2(2r^2-x^2)}{\sqrt{4r^2-x^2}}$

For maximum/minimum area, $\dfrac{dA}{dx} = 0 \Rightarrow \dfrac{2r^2-x^2}{\sqrt{4r^2-x^2}} = 0 \Rightarrow x = \sqrt{2}r$

Now, $\dfrac{d^2A}{dx^2} = \dfrac{\sqrt{4r^2-x^2}(-4x)-(4r^2-2x^2)\dfrac{1\times(-2x)}{2\sqrt{4r^2-x^2}}}{(4r^2-x^2)}$

$= \dfrac{(4r^2-x^2)(-4x)+(4r^2-2x^2)x}{(4r^2-x^2)^{3/2}} = \dfrac{-12r^2x+2x^3}{(4r^2-x^2)^{3/2}}$

$\left(\dfrac{d^2A}{dx^2}\right)_{x=\sqrt{2}r} = \dfrac{-12r^2(\sqrt{2}r)+2(\sqrt{2}r)^3}{(2r^2)^{3/2}}$

$= \dfrac{4\sqrt{2}r^3-\sqrt{2}r^3}{2\sqrt{2}r^3} = 2-6 < 0$ $\therefore$ Area is maximum at $x = \sqrt{2}r$

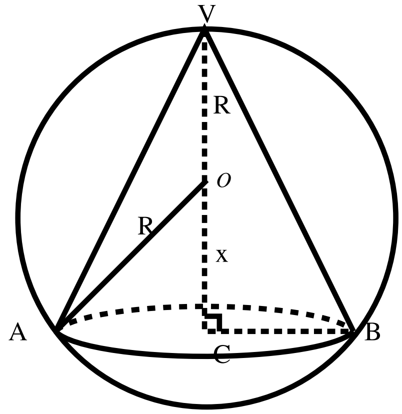
$\therefore$ Length of rectangle is $\sqrt{2}r$ and width of rectangle is $\sqrt{4r^2-2r^2}=\sqrt{2}r$

Hence, of all the rectangles the square of side $\sqrt{2}r$ units has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that height is equal to the diameter of the base.

SOLUTION

Let r be the radius of the circular base, h be the height and S be the total surface area of a right circular cylinder, then $S = 2\pi r^2 + 2\pi rh$



Let V be the volume of cylinder with the above dimensions, then $V=\pi r^2 h=\pi r^2\left(\frac{S-2\pi r^2}{2\pi r}\right)=\frac{r}{2}(S-2\pi r^2)$

$\Rightarrow V=\frac{Sr}{2}-\pi r^3$ (i)

Differentiating (i) w.r.t. r, we get $\frac{dV}{dr}=\frac{S}{2}-3\pi r^2$

For maximum/minimum volume, $\frac{dV}{dr}=0\Rightarrow\frac{S}{2}-3\pi r^2=0\Rightarrow r^2=\frac{S}{6\pi}\Rightarrow r=\sqrt{\frac{S}{6\pi}}$

$\frac{d^2V}{dr^2}=-6\pi r$ and $\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{S/(6\pi)}}=-6\pi\sqrt{\frac{S}{6\pi}}<0$

$\therefore$ V has a maximum value at $r=\sqrt{\frac{S}{6\pi}}$. When $r=\sqrt{\frac{S}{6\pi}}$, then $h=\frac{S-2\pi r^2}{2\pi r}=\frac{S-2\pi\left(\frac{S}{6\pi}\right)}{2\pi\sqrt{\frac{S}{6\pi}}}=\frac{4\pi S/6\pi}{2\pi\sqrt{\frac{S}{6\pi}}}$

$\Rightarrow h=2\sqrt{\frac{S}{6\pi}}=2$ (radius)=diameter So, volume is maximum when the height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, of the dimensions of the can which has the minimum surface area.

SOLUTION

Let r cm be the radius, h cm be the height, $S cm^2$ be the total surface area and $V cm^3$ be the volume, then $V = \pi r^2 h = 100 \Rightarrow h = \dfrac{100}{\pi r^2}$ (i) and $S = 2\pi r^2 + 2\pi rh$ (ii)

$\Rightarrow S=2\pi r^2+2\pi r\left(\frac{100}{\pi r^2}\right)=2\pi r^2+\frac{200}{r}$ (using (i))

Differentiating $S=2\pi r^2+\frac{200}{r}$ w.r.t. r, we get $\frac{dS}{dr}=4\pi r-\frac{200}{r^2}$

For maximum/minimum surface area, $\frac{dS}{dr}=0\Rightarrow 4\pi r-\frac{200}{r^2}=0\Rightarrow r^3=\frac{200}{4\pi}\Rightarrow r=\left(\frac{50}{\pi}\right)^{1/3}$

$\frac{d^2S}{dr^2}=4\pi+\frac{200\times 2}{r^3}=4\pi+\frac{400}{r^3}$ and $\left(\frac{d^2S}{dr^2}\right)_{r=(50/\pi)^{1/3}}=4\pi+\frac{400}{(50/\pi)}>0\therefore S$ has a minimum, when $r=\left(\frac{50}{\pi}\right)^{1/3}$

When $r=\left(\frac{50}{\pi}\right)^{1/3}$ $h=\frac{100}{\pi r^2}=\frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}}=\frac{100}{(50)^{2,3}\pi^{1/3}}=\frac{50\times 2}{(50)^{2/3}\pi^{1/3}}=2\left(\frac{50}{\pi}\right)^{1/3}$

Therefore, radius $=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $=2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

22. A wire of length 28m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum?

SOLUTION

Let the length of the piece bent into the shape of a circle be xm and so length of the other piece bent into the shape of a square is $(28-x)$m.

Circumference of circle $=2\pi r\Rightarrow 2\pi r=x\Rightarrow r=\frac{x}{2\pi}\Rightarrow$ Area of the circle$=\pi(r)^2=\pi\left(\frac{x}{2\pi}\right)^2=\frac{x^2}{4\pi}$.

Perimeter of square $=4(\text{side})\Rightarrow 28-x=4(\text{side})\Rightarrow \text{Side}=\frac{28-x}{4}$

$\Rightarrow$ Area of the square $=(\text{side})^2=\left(\frac{28-x}{4}\right)^2=\frac{(28-x)^2}{16}$.

Let A be the sum of the areas of the two shapes, then $A=\frac{x^2}{4\pi}+\frac{(28-x)^2}{16}$ (i) Differentiating (i) w.r.t. x, we get $\frac{dA}{dx}=\frac{2x}{4\pi}+\frac{2(28-x)(-1)}{16}=\frac{x}{2\pi}-\frac{28-x}{8}$

For maximum/minimum area, $\frac{dA}{dx}=0\Rightarrow\frac{x}{2\pi}-\frac{28-x}{8}=0\Rightarrow\frac{4x-28\pi+x\pi}{8\pi}=0$

$\Rightarrow 4x+x\pi=28\pi=x=\frac{28\pi}{4+\pi}$

$\frac{d^2A}{dx^2}=\frac{1}{2\pi}-\frac{(-1)}{8}=\frac{1}{2\pi}+\frac{1}{8}$ and $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{28\pi}{4+\pi}}=\frac{1}{2\pi}+\frac{1}{8}>0$

Hence, area A is minimum at $x=\frac{28\pi}{4+\pi}\therefore$ The wire must be cut at a distance of $\frac{28\pi}{4+\pi}$m from one end.

Hence, the lengths of the two pieces are $28-\frac{28\pi}{4+\pi}=\frac{112}{4+\pi}$m and $\frac{28\pi}{4+\pi}$m.

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $8/27$ of the volume of the sphere.

SOLUTION

Let PAB be a cone of greatest volume inscribed in the sphere. Let $OC=x$. Then in $\Delta OAC, AC=\sqrt{R^2-x^2}$ and $PC=PO+OC=R+x=$height of the cone.

Let V be the volume of the cone. Then, $V=\frac{1}{3}\pi(AC)^2(PC)\Rightarrow V=\frac{1}{3}\pi(R^2-x^2)(R+x)$

$= \frac{1}{3}\pi(R^3 + xR^2 - Rx^2 - x^3)$ (i)

Differentiating (i) w.r.t. x, we get $\frac{dV}{dx} = \frac{1}{3}\pi\left[(R^2 - x^2) - 2x(R + x)\right] \Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi(R^2 - 2Rx - 3x^2)$

For maximum/minimum V we must have $\frac{dV}{dx} = 0 \Rightarrow R^2 - 2Rx - 3x^2 = 0 \Rightarrow (R - 3x)(R + x) = 0$

$\Rightarrow R - 3x = 0 \Rightarrow x = \frac{R}{3}\ \frac{d^2V}{dx^2} = \frac{1}{3}\pi(-2R - 6x)$ and

$\left(\frac{d^2V}{dx^2}\right)_{x=R/3} = -\frac{4}{3}R\pi < 0$

Thus, V is maximum when $x = \frac{R}{3}$. Putting $x = \frac{R}{3}$ in $V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$, we obtain

Maximum volume of the cone $= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right) \Rightarrow$ Maximum volume of the cone $= \frac{32\pi R^3}{81}$

$\Rightarrow$ Maximum volume of the cone

$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$

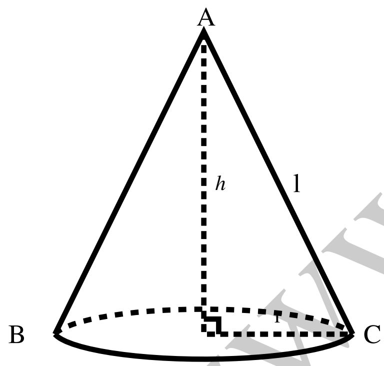
$= \frac{8}{27}$ (Volume of the sphere) Hence proved.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

SOLUTION

Let r and h be the radius and height respectively of the cone ABC

Volume, $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$



Curved surface area, $S = \pi r l = \pi r\left(\sqrt{h^2 + r^2}\right) = \pi r\left(\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right)$ $S^2 = (\pi r)^2\left[\left(\frac{3V}{\pi r^2}\right)^2 + r^2\right] = \frac{(3V)^2}{r^2} + \pi^2 r^4$

Let $Z = S^2 \Rightarrow Z = \frac{(3V)^2}{r^2} + \pi^2 r^4$ (i) Differentiating (i) w.r.t. r, we get $\frac{dZ}{dr} = \frac{-2(3V)^2}{r^3} + 4\pi^2 r^3$

For maximum/minimum surface area, $\frac{dZ}{dr} = 0$

$\Rightarrow \frac{-2(3V)^2}{r^3} + 4\pi^2 r^3 = 0 \Rightarrow -2(3V)^2 + 4\pi^2 r^6 = 0$

$\Rightarrow -(3V)^2 + 2\pi^2 r^6 = 0 \Rightarrow 2\pi^2 r^6 = (3V)^2 \Rightarrow r^6 = \frac{(3V)^2}{2\pi^2} = \frac{\left(3\left(\frac{1}{3}\pi r^2 h\right)\right)^2}{2\pi^2} = \frac{r^4 h^2}{2} \Rightarrow r^2 = \frac{h^2}{2} \Rightarrow 2r^2 = h^2$

$\frac{d^2Z}{dr^2} = -2(3V)^2\left[\frac{-3}{r^4}\right] + 12\pi^2 r^2$

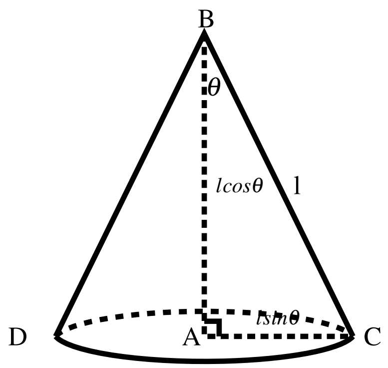
and $\left(\frac{d^2Z}{dr^2}\right)_{r^2=\frac{h^2}{2}} > 0$

Hence, surface area is minimum at $2r^2 = h^2 \Rightarrow h = \sqrt{2}r$

25. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

SOLUTION

Let θ be the semi vertical angle, l be the given slant height, then radius of base $(r) = l\sin\theta$, height$(h) = l\cos\theta$ (is right angled triangle) and volume of cone$=\frac{1}{3}\pi r^2 h$



$\Rightarrow V = \frac{1}{3}\pi(l\sin\theta)^2 l\cos\theta = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$

$\Rightarrow \frac{dV}{d8} = \frac{1}{3}\pi l^3\left\{\left(\sin^2\theta\right)(-\sin\theta) + \cos\theta\times 2\sin\theta cos\theta\right\}$

$= \frac{1}{3}\pi l^3\sin\theta\left[-\sin^2\theta + 2\left(1-\sin^2\theta\right)\right] = \frac{1}{3}\pi l^3\sin\theta\cos^2\theta\left[2\sec^2\theta - 3\tan^2\theta\right]$

$= \frac{1}{3}\pi l^3\sin\theta\cos^2\theta\left[2-\tan^2\theta\right]$ For maximum/minimum volume,

$\frac{dV}{d\theta} = 0 \Rightarrow \frac{1}{3}\pi l^3\sin\theta\cos^2\theta\left(2-\tan^2\theta\right) = 0 \Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \left(\sqrt{2}\right) \Rightarrow \theta = \tan^{-1}\left(\sqrt{2}\right)$

$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3\cos^3\theta\left(2-7\tan^2\theta\right) \Rightarrow \left(\frac{d^2V}{d\theta^2}\right)_{\tan\theta=\sqrt{2}} = \frac{1}{3}\pi l^3\left(\frac{1}{\sqrt{3}}\right)^3(2-7\times 2) = -\frac{4\pi l^3}{3\sqrt{3}} < 0$

Thus, V is maximum, when $\tan\theta = \sqrt{2}$ or $\theta = \tan^{-1}\sqrt{2}$ Thus, the semi-vertical angle of the cone is $\tan^{-1}\sqrt{2}$.

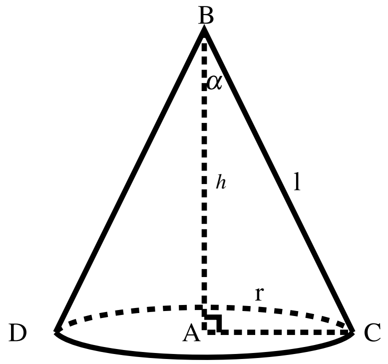
26. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

. SOLUTION

Let r be the radius, l be the slant height and h be the vertical height and α be the semi-vertical angle of cone of given surface area S. Then,

$S = \pi rl + \pi r^2 \Rightarrow l = \frac{S-\pi r^2}{\pi r}$

Let V be the volume of the cone. $\therefore V = \frac{1}{3}\pi r^2 h \Rightarrow V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4\left(l^2 - r^2\right)$



$$\Rightarrow V^2 = \frac{\pi^2}{9}r^4\left[\left(\frac{S-\pi r^2}{\pi r}\right)^2 - r^2\right] = \frac{\pi^2 r^4}{9}\left[\frac{(S-\pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$\Rightarrow V^2 = \frac{1}{9}S(Sr^2 - 2\pi r^4)$$

Let $Z = V^2$, then y is maximum or minimum according as Z is maximum or minimum. $\therefore Z = \frac{1}{9}S\left(Sr^2 - 2\pi r^4\right) \Rightarrow \frac{dZ}{dr} = \frac{1}{9}S\left(2Sr - 8\pi r^3\right)$

For maximum/minimum Z, we have $\frac{dZ}{dr} = 0$

$$\Rightarrow 2Sr - 8\pi r^3 = 0 \Rightarrow S = 4\pi r^2 \Rightarrow r = \sqrt{\frac{S}{4\pi}}$$

Now, $\frac{d^2Z}{dr^2} = \frac{S}{9}(2S - 24\pi r^2)$

$$\Rightarrow \left(\frac{d^2Z}{dr^2}\right)_{r=\sqrt{\frac{S}{4\pi}}} = \frac{S}{9}\left(2S - 24\pi\frac{S}{4\pi}\right)$$

$$\Rightarrow \frac{d^2Z}{dr^2} = -\frac{4S^2}{9} < 0$$

So, Z is maximum when $S = 4\pi r^2$.

Hence, V is maximum when $S = 4\pi r^2$

Now, $S = 4\pi r^2 \Rightarrow \pi rl + \pi r^2 = 4\pi r^2 \Rightarrow l = 3r$ In ΔAOC

$$\sin\alpha = \frac{r}{l} = \frac{r}{3r} = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\frac{1}{3}$$

Hence V is maximum when $\alpha = \sin^{-1}\frac{1}{3}$.

Choose the correct answer in the Exercise from 27 to 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point $(0,5)$ is

(A) $(2\sqrt{2},4)$

(B) $(2\sqrt{2},0)$

(C) $(0,0)$

(D) $(2,2)$

SOLUTION

(A): Let Z be the square of distance of the point $P(x,y)$ on $x^2 = 2y$ from the point $A(0,\ 5)$, then $Z = |PA|^2$

$$\Rightarrow Z = (x-0)^2 + (y-5)^2 = 2y + (y-5)^2 = y^2 - 8y + 25 \Rightarrow \frac{dZ}{dy} = 2y - 8$$

Now, $\frac{dZ}{dy}=0 \Rightarrow 2y-8=0 \Rightarrow y=4 \therefore x^2=2(4)=8$[From $x^2=2y$]

$\Rightarrow x=2\sqrt{2}$ $\left(\frac{d^2Z}{dy^2}\right)_{y=4}=2>0$ Hence, Z is minimum at $(2\sqrt{2},4)$

.

28. For all real values of$x$, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is (A) 0

(B) 1

(C) 3

(D) $(1/3)$

SOLUTION

(D):

Let $y=\frac{x^2-x+1}{x^2+x+1} \Rightarrow \frac{dy}{dx}=\frac{2(x-1)(x+1)}{(x^2+x+1)^2}$

For critical points, we have $\frac{dy}{dx}=0 \Rightarrow x=1,-1$

At $x=1$, $\frac{dy}{dx}$ changes its sign from $-ve$ to $+ve$.

At $x=-1$, $\frac{dy}{dx}$ changes its sign from $+ve$ to $-ve$.

Thus, y is minimum at $x=1$. $\therefore$ Minimum value of $y=\frac{1-x+x^2}{1+x+x^2}$ at $x=1$ is $\frac{1-1+1}{1+1+1}=\frac{1}{3}$.

29. The maximum value of $[x(x-1)+1]^{1/3}, 0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{1/3}$

(B) $\frac{1}{2}$

(C) 1

(D) 0

SOLUTION

(C): Let $f(x)=[x(x-1)+1]^{1/3}=(x^2-x+1)^{1/3}, 0 \le x \le 1$

$\Rightarrow f'(x)=\frac{1}{3}(x^2-x+1)^{\frac{1}{3}-1}(2x-1)=\frac{1}{3}(x^2-x+1)^{\frac{-2}{3}}(2x-1)$

For critical points, $f'(x)=0 \Rightarrow 2x-1=0=x=\frac{1}{2} \in [0,1]$

For maximum or minimum value of f, we evaluate $f(0), f(1)$ and $f(1/2)$.

Here, $f(0)=(0-0+1)^{1/3}=1, f(1)=(1(0)+1)^{1/3}=1$ and $f\left(\frac{1}{2}\right)=\left(\frac{1}{4}-\frac{1}{2}+1\right)^{1/3}=\left(\frac{3}{4}\right)^{1/3}$

Maximum value of $f(x)$ is 1.