NCERT - Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

   SOLUTION

   We have, $y = 3x^4 - 4x$ (i)

   Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = 3 \cdot 4x^3 - 4 \cdot 1 = 12x^3 - 4 \therefore$ Slope of tangent at $x = 4$ is $\left(\frac{dy}{dx}\right)_{x=4} = 12 \times (4)^3 - 4 = 764$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

   SOLUTION

   We have, $y = \frac{x-1}{x-2}, x \neq 2$ (i) Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = \frac{(x-2) \cdot 1 - (x-1) \cdot 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$

   $\therefore$ Slope of tangent at $x = 10$ is $\left(\frac{dy}{dx}\right)_{x=10} = \frac{-1}{(10-2)^2} = -\frac{1}{64}$

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose $x-$coordinate is 2.

   SOLUTION

   We have, $y = x^3 - x + 1$ (i)

   Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = 3x^2 - 1 \therefore$ Slope of tangent at $x = 2$ is $\left(\frac{dy}{dx}\right)_{x=2} = 3(2)^2 - 1 = 11$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose $x-$coordinate is 3.

   SOLUTION

   We have, $y = x^2 - 3x + 2$ (i)

   Differentiating (i) w.r.t x, we get $\frac{dy}{dx} = 3x^2 - 3 \therefore$ Slope of tangent at $x = 3$ is $\left(\frac{dy}{dx}\right)_{x=3} = 3 \times 3^2 - 3 = 24.$

5. Find the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$at $\theta = \frac{\pi}{4}$.

   SOLUTION

   We have $x = a\cos^3\theta$ (i) $y = a\sin^3\theta$ (ii)

   Differentiating (i) & (ii) w.r.t θ,we get $\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$

   $$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

   $\therefore$ Slope of normal at $\theta = \frac{\pi}{4}$ is $\frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}}} = \frac{-1}{-\tan(\pi/4)} = 1$

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta, y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

   SOLUTION

   We have $x = 1 - a\sin\theta$ and (i) $y = b\cos^2\theta$

Differentiating (i) & (ii) w.r.t θ,we get $\frac{dx}{d\theta} = 0 - a\cos\theta = -a\cos\theta$ and

$\frac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$ So, $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2bcos\theta(\sin\theta)}{-a\cos\theta} = \frac{2b}{a}\sin\theta$

$\therefore$ Slope of normal at $\theta = \frac{\pi}{2}$ is $\frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\pi/2}}$

$= \frac{-1}{\frac{2b}{a}sin\left(\frac{\pi}{2}\right)} = \frac{-a}{2b}$

7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the $x-$axis.

SOLUTION

We have, $y = x^3 - 3x^2 - 9x + 7$ (i)

Differentiating (i) w.r.t x, we get $\frac{dy}{dx} = 3x^2 - 6x - 9$

Now, tangent to (i) is parallel to $x-$axis$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3, -1$

When $x = 3$, the from (i), we get $y = 3^3 - 3\cdot(3^2) - 9\cdot 3 + 7 = 27 - 27 - 27 + 7 = -20$

When $x = -1$, then from (i), we get $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$

Hence, the required points are $(3, -20)$ and $(-1, 12)$.

8. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$.

SOLUTION

Equation of given curve is $y = (x-2)^2$ (i) $\Rightarrow \frac{dy}{dx} = 2(x-2)$ Slope of chord joining the points $(2,0)$ and $(4,4)$ is $\frac{4-0}{4-2} = \frac{4}{2} = 2$

For the points at which tangent is parallel to the chord joining points $(2,0)$ and $(4,4)$, we must have $\frac{dy}{dx} =$ slope of the chord $\Rightarrow 2(x-2) = 2 \Rightarrow x - 2 = 1 \Rightarrow x = 3$

When $x = 3$, then from (i), we get $y = (3-2)^2 = 1$ $\therefore$ Required point is $(3,1)$

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

SOLUTION

We have, $y = x^3 - 11x + 5$ (i) and $y = x - 11$ (ii)

Slope of (ii) is 1 (iii)

From (i), $\frac{dy}{dx} = 3x^2 - 11$ Slope of tangent is $\frac{dy}{dx} = 1$ [from(iii)] $\Rightarrow 3x^2 - 11 = 1 \Rightarrow x = \pm 2$

When $x = 2$, then from (i), $y = 2^3 - 11\times 2 + 5 = -9$ When $x = -2$, then from (i), $y = (-2)^3 - 11(-2) + 5 = 19$

So, we find that at $(2, -9)$and at $(-2, 19)$ the slope of tangent is 1.

But only $(2, -9)$ satisfies given equation of tangent. $\therefore$ The point at which the line (ii) is tangent is $(2, -9)$.

10. Find the equations of all lines having slope -1 that are tangents to the curve $y = \dfrac{1}{x-1}$, $x \neq 1$.

SOLUTION

We have, $y=\dfrac{1}{x-1}, x\neq 1$ (i)

Differentiating (i) w.r.t. x, we get $\dfrac{dy}{dx}=\dfrac{-1}{(x-1)^2}$

For tangents having slope $=-1$, we must have-l $=\dfrac{-1}{(x-1)^2}\Rightarrow(x-1)^2=1\Rightarrow x-1=\pm1\Rightarrow x=1\pm1=2,0$

When $x=2$, then from (i), $y=\dfrac{1}{2-1}=1\therefore$ The point is $(2,\ 1)$.

Equation of tangent at $(2,\ 1)$ is $y-1=-1(x-2)$, or $x+y-3=0$ When $x=0$, then from (i), $y=\dfrac{1}{0-1}=-1\therefore$ The point is $(0,-1)$. Equation of tangent at $(0,\ -1)$ is $y-(-1)=-1(x-0)$, or $x+y+1=0$

$\therefore$ Required tangents are $x+y-3=0$ and $x+y+1=0$.

11. Find the equations of all lines having slope 2 which are tangents to the curve $y=\dfrac{1}{x-3}, x\neq 3$.

SOLUTION

The given curve is $y=\dfrac{1}{x-3}$ (i)

Differentiating (i) w.r.t. x, we get $\dfrac{dy}{dx}=\dfrac{-1}{(x-3)^2}$

For tangents having slope 2, we must have $2=\dfrac{-1}{(x-3)^2}\Rightarrow(x-3)^2=-\dfrac{1}{2}\Rightarrow 2(x-3)^2=-1\Rightarrow 2x^2-12x+19=0$

$\Rightarrow x=\dfrac{12\pm\sqrt{144-152}}{4}\Rightarrow x=\dfrac{12\pm\sqrt{-8}}{4}$ which is not possible as being imaginary number.
Hence, there is no tangent.

12. Find the equations of all lines having slope 0 which are tangents to the curve $y=\dfrac{1}{x^2-2x+3}$.

SOLUTION

We have, $y=\dfrac{1}{x^2-2x+3}$ (i)

Differentiating (i), w.r.t. x, we get $\dfrac{dy}{dx}=\dfrac{-1}{(x^2-2x+3)^2}\dfrac{d}{dx}(x^2-2x+3)=\dfrac{-(2x-2)}{(x^2-2x+3)^2}$

For tangents having slope 0, we must have $\dfrac{dy}{dx}=0\Rightarrow\dfrac{-(2x-2)}{(x^2-2x+3)}=0\Rightarrow 2x-2=0\Rightarrow x=1$

When $x=1, y=\dfrac{1}{1^2-2\cdot1+3}=\dfrac{1}{2}$ (using (i)) $\therefore$ The tangent to the curve (i) at $\left(1,\ \dfrac{1}{2}\right)$ with slope 0 will be given by $y-\dfrac{1}{2}=0(x-1)$, or $2y-1=0$, or $y=\dfrac{1}{2}$

13. Find points on the curve $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$ at which the tangents are

(i) parallel to x-axis

(ii) parallel to y-axis.

SOLUTION

We have, $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$ (1)

Differentiating (1) w.r.t. x, we get $\dfrac{2x}{9}+\dfrac{1}{16}\left(2y\dfrac{dy}{dx}\right)=0\Rightarrow\dfrac{y}{8}\dfrac{dy}{dx}=-\dfrac{2x}{9}$

$\Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$ (2)

(i) For tangents parallel to x-axis, we must have $\frac{dy}{dx} = 0 \Rightarrow -\frac{16x}{9y} = 0 = x = 0$

When $x = 0$, then from (1), $\frac{0^2}{9} + \frac{y^2}{16} = 1 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$

$\therefore$ The points on (1) at which the tangents are parallel to x-axis are $(0, \dot{4})$ and $(0,\ -4)$. (ii) For tangents parallel to y-axis, we must have $\frac{dx}{dy} = 0$

$\Rightarrow -\frac{9y}{16x} = 0 \Rightarrow y = 0$

When $y = 0$, then from (1), $\frac{x^2}{9} + \frac{0^2}{16} = 1 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$\therefore$ The points on (1) at which the tangents are parallel to y-axis are$(3,\ 0)$ and $(-3, 0)$.

14. Find the equations of the tangent and normal to the given curves at the indicated points: (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$
(ii) $y = x^4 - d + 13x^2 - 10x + 5$ at $(1,\ 3)$
(iii) $y = x^3$ at $(1, 1)$
(iv) $y = x^2$ at $(0, 0)$
(v) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$.

SOLUTION

(i) We have, $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

(1) Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$ $\therefore$ Slope of tangent at $(0, 5)$ is $\left(\frac{dy}{dx}\right)_{(0,5)} = -10$

So, equation of the tangent to (1) at $(0, 5)$ is $y - 5 = -10(x - 0)$, or $10x + y - 5 = 0$

Again, the slope of normal at $(0, 5) = \frac{-1}{\text{Slope of tangent}} = \frac{-1}{-10} = \frac{1}{10}$.

So, equation of the normal to (1) at $(0, 5)$ is $y - 5 = \frac{1}{10}(x - 0)$, or $x - 10y + 50 = 0$

(ii) We have, $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ (1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$ $\therefore$ Slope of tangent at $(1,\ 3)$ is $\left(\frac{dy}{dx}\right)_{(1,3)} = 4 - 18 + 26 - 10 = 2$

So, equation of the tangent to (1) at $(1,\ 3)$ is $y - 3 = 2(x - 1)$ or $2x - y + 1 = 0$

Again, the slope of normal at $(1,\ 3) = \frac{-1}{\text{Slope of tangent}} = \frac{-1}{2}$

Hence, the equation of the normal to (1) at $(1,\ 3)$ is $y - 3 = -\frac{1}{2}(x - 1)$, or $x + 2y - 7 = 0$ (1)

(iii) We have, $y = x^3$ (1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = 3x^2$

So, slope of the tangent to (1) at $(1,\ 1)$ is $\left(\frac{dy}{dx}\right)_{(1,1)} = 3(1)^2 = 3$ $\therefore$ The equation of the tangent to (1) at $(1,\ 1)$ is $y - 1 = 3(x - 1)$

or $3x - y - 2 = 0$ Again, the slope of normal at $(1,\ 1) = \frac{-1}{\text{Slope of tangent}} = \frac{-1}{3}$

Hence, the equation of the normal to(1) at $(1, 1)$is $y - 1 = -\frac{1}{3}(x - 1)$ or $x + 3y - 4 = 0$

(iv) The given curve is $y = x^2$ (1) Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = 2x$

The slope of the tangent to (1) at $(0,0) = \left(\frac{dy}{dx}\right)_{(0,0)} = 2 \times 0 = 0$

So , the equation of the tangent to (1) at $(0,0)$ is $y - 0 = 0(x - 0)$ or $y = 0$

The equation of the normal line to (1) at $(0,0)$ is $(y - 0) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(0,0)}}(x - 0)$

$y\left(\frac{dy}{dx}\right)_{(0,0)} = -x \Rightarrow y(0) = -x \Rightarrow x = 0$

Alternately, tangent at $(0,0)$ is parallel to x-axis, therefore, normal to (1) at $(0,0)$is parallel to y-axis and its equation is $x = 0$. (Line through $(0,0)$ and parallel to y-axis is $x = 0$)

(v) We have, $x = \cos t$, $y = \sin t$ (1) $\Rightarrow \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

The point on the curve at $t = \frac{\pi}{4}$ is $\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right)$ or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Also, the slope of the tangent at $t = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$

$\therefore$ Slope of the tangent to (1) at $t = \frac{\pi}{4}$is $-\cot(\pi/4) = -1$.

So, the equation of the tangent to (1) at $t = \frac{\pi}{4}$ is $y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$ or $x + y - \frac{2}{\sqrt{2}} = 0$ or $x + y - \sqrt{2} = 0$

Also, the slope of the normal to (1) at $t = \frac{\pi}{4}$ is $\frac{-1}{\text{Slope of tangent}} = \frac{-1}{-1} = 1$

$\therefore$ The equation of the normal to (1) at $t = \frac{\pi}{4}$ is $y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right)$ or $x - y = 0$

15. . Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$, which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

SOLUTION

We have, $y = x^2 - 2x + 7$ (i) Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = 2x - 2$

(a) The slope of the tangent to the curve (i) is $2x - 2$ Slope of line $2x + y - 9 = 0$ is 2 Since the tangent is parallel to the line $2x + y - 9 = 0$, $\therefore$ Their slopes are equal $\Rightarrow 2x - 2 = 2 \Rightarrow x = 2$ Putting $x = 2$in (i),we get $y = 7$.

$\therefore$ The equation of tangent at $(2,7)$ parallel to $2x + y - 9 = 0$is $(y - 7) = 2(x - 2) \Rightarrow y - 7 = 2x - 4 \Rightarrow 2x - y + 3 = 0$

(b) S1ope of tangent to curve (i) is $2x - 2$ Slope of line $5y - 15x = 13$ is 3

Since the required tangent is perpendicular to the line $5y - 15x = 13$ $\therefore$ Product of their slopes is $-1 \Rightarrow (2x - 2)(3) = -1 \Rightarrow 6x - 6 = -1 \Rightarrow 6x = 5 \Rightarrow x = \frac{5}{6}$

Putting $x = \frac{5}{6}$ in (i), we get $y = \frac{217}{36}$

Also, slope of the required tangent $= \frac{-1}{3}$

$\therefore$ The equation of tangent at $(\frac{5}{6}, \frac{217}{36})$ perpendicular to $5y - 15x = 13$ is $\left(y - \frac{217}{36}\right) = -\frac{1}{3}\left(x - \frac{5}{6}\right) \Rightarrow 12x + 36y - 227 = 0$

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

SOLUTION

We have, $y = 7x^3 + 11$ (i)

Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = 7 \cdot (3x^2) + 0 = 21x^2$

$\therefore$ Slope of tangent at $x = 2$ is $\left(\frac{dy}{dx}\right)_{x=2} = 21(2)^2 = 84$ and slope of tangent at $x = -2$ is $\left(\frac{dy}{dx}\right)_{x=-2} = 21(-2)^2 = 84$

Hence, the slopes of tangents at $x = 2$ and $x = -2$ are equal. Therefore, these tangents are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

SOLUTION

We have, $y = x^3$ (i)

Differentiating (i), w.r.t. x , we get $\frac{dy}{dx} = 3x^2$ Since, it is given that slope is equal to the $y-$coordinate of the point

$\therefore \frac{dy}{dx} = y \Rightarrow 3x^2 = y$ (using (ii)) $\Rightarrow 3x^2 = x^3 \Rightarrow x^2(3 - x) = 0 \Rightarrow x = 0$ or $x = 3$ (using (i))

When $x = 0$, then from (i)$y = 0$ When $x = 3$, then from (i), $y = 3^3 = 27$ $\therefore$ The required points are $(0,0)$ and $(3,\ 27)$.

18. For the curve$y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

SOLUTION

Let (x_1, y_1) be the required point on the given curve $y = 4x^3 - 2x^5$ (i) $\therefore y_1 = 4{x_1}^3 - 2{x_1}^5$

Differentiating (i) w. r. t. x, we get $\frac{dy}{dx} = 12x^2 - 10x^4$

So, $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 12{x_1}^2 - 10{x_1}^4$ $\therefore$ The equation of the tangent at (x_1, y_1) is $y - y_1 = (12x_1^2 - 10{x_1}^4)(x - x_1)$

Since, it passes through the origin,

$\therefore 0 - y_1 = (12x_1^2 - 10{x_1}^4)(0 - x_1)$ or $y_1 = 12x_1^3 - 10x_1^5$ (iii)

From (ii) and (iii), $4{x_1}^3 - 2x_1^5 = 12x_1^3 - 10{x_1}^5 \Rightarrow -8x_1^3 + 8x_1^5 = 0 \Rightarrow 8x_1^3(-1 + x_1^2) = 0 \Rightarrow x_1 = 0, x_1 = \pm 1$

When $x_1 = 0$, then from(ii),$y_1 = 0$ When $x_1 = 1$, then from(ii),$y_1 = 4(1) - 2(1) = 2$

When $x_1 = -1$, then from(ii),$y_1 = 4(-1)^3 - 2(-1)^5 = -4 + 2 = -2$

Hence, the required points are$(0,0), (1,2)$and$(-1,-2)$

19. . Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the $x-$ axis.

SOLUTION

$x^2 + y^2 - 2x - 3 = 0$ (i)

Differentiating (i) w. r.t. x, we get $2x + 2y\frac{dy}{dx} - 2 - 0 = 0 \Rightarrow \frac{dy}{dx} = \frac{2(1-x)}{2y} = \frac{1-x}{y}$ (ii)

For tangents parallel to x-axis, we must have $\frac{dy}{dx} = 0 \Rightarrow \frac{1-x}{y} = 0 \Rightarrow x = 1, y \neq 0$

Substituting $x = 1$ in (i), we get $1^2 + y^2 - 2 \cdot 1 - 3 = 0 \Rightarrow y^2 - 4 = 0 \Rightarrow y = \pm 2$

Hence, the required points are $(1,2)$ and $(1,-2)$.

20. Find the equation of the normal at the point $\left(am^2, am^3\right)$ for the curve $ay^2 = x^3$.

SOLUTION

We have, $ay^2 = x^3$ (i)

Differentiating (i) w. r.t. x, we get $a(2y)\frac{dy}{dx}=3x^2 \Rightarrow \frac{dy}{dx}=\frac{3x^2}{2ay}$ $\therefore$ Slope of tangent at $\left(am^2, am^3\right)$

$=\left(\frac{dy}{dx}\right)_{\left(am^2,am^3\right)}=\frac{3(am^2)^2}{2a(am^3)}=\frac{3}{2}m \Rightarrow$ Slope of normal at the given point$=-\frac{1}{\frac{3}{2}m}=-\frac{2}{3m}$

Hence the equation of normal at the given point is $\left(y-am^3\right)=-\frac{2}{3m}\left(x-am^2\right)$ or $3my-3am^4=-2x+2am^2$ or $2x+3my-3am^4-2am^2=0$

21. Find the equation of the normals to the curve $y=x^3+2x+6$ which are parallel to the line $x+14y+4=0$.

SOLUTION

We have $y=x^3+2x+6$ (i)

The given line is $14y+x+4=0$ (ii)

The slope of line (ii) is $-\frac{1}{14}$.

Differentiating (i) w.r.t. x, we get $\frac{dy}{dx}=3x^2+2$

Let $P(x_1,\ y_1)$ be a point on (i). $\therefore$ The slope of tangent at $P(x_1,y_1)$ to (i) is

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=3{x_1}^2+2$

$\Rightarrow$ The slope of normal at $P(x_1,y_1)$ to (i) is $-\frac{1}{3x_1^2+2}$

Since normal at $P(x_1,y_1)$ to (i) is parallel to the line (ii), we get $-\frac{1}{3x_1^2+2}=-\frac{1}{14} \Rightarrow 3{x_1}^2+2=14 \Rightarrow 3{x_1}^2=12 \Rightarrow x_1^2=4 \Rightarrow x_1=\pm 2$

As $P(x_1,y_1)$ lies on the curve (i) , we get $y_1={x_1}^3+2x_1+6$and so when $x_1=2, y_1=2^3+2\cdot 2+6=18$ and when $x_1=-2, y_1=(-2)^3+2\cdot(-2)+6=-6$

Thus, there are two points $(2,\ 18)$ and $(-2,\ -6)$ on (i) at which the normals are parallel to (ii).

Therefore, the equations of the required normals are $y-18=-\frac{1}{14}(x-2)$ and

$y+6=-\frac{1}{14}(x+2)$ or, $14y-252=-x+2$ and $14y+84=-x-2$ or, $x+14y-254=0$and $x+14y+86=0$

22. Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point $(at^2,2at)$

SOLUTION

We have, $y^2=4ax$ (i)

Differentiating (i) w.r.t. x, we get $2y\frac{dy}{dx}=4a \Rightarrow \frac{dy}{dx}=\frac{2a}{y}$ $\therefore$ Slope of the tangent at

$(at^2,2at)$ is $\left(\frac{dy}{dx}\right)_{(at^2,2at)}=\frac{2a}{2at}=\frac{1}{t}$

Hence, the equation of the tangent to (i) at $(at^2,\ 2at)$ is $y-2at=\frac{1}{t}(x-at^2)$ or $x-ty+at^2=0$

Slope of normal at $(at^2,2at)$ is $\frac{-1}{\text{Slope of tangent}}=-t$

$\therefore$ The equation of the normal to (i) at $(at^2,\ 2at)$ is $y-2at=-t(x-at^2)$ or $tx+y-2at-at^3=0$

23. Prove that the curves $x=y^2 and\ xy=k$ cut at right angles, if $8k^2=1$.

SOLUTION

We have, $x=y^2$ (i) and $xy=k$ (ii)

Solving (i) &(ii), we get $y^3 = k \Rightarrow y = k^{1/3}$ Substituting this value of y in (i), we get $x = (k^{1/3})^2 = k^{2/3}$ $\therefore$ (i) and (ii) intersect at the point $(k^{2/3},\ k^{1/3})$.

Differentiating (i) w.r.t. x, we get $1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

Slope of tangent to (i) at $(k^{2/3},\ k^{1/3}) = \frac{1}{2k^{1/3}}$ (iii) From (ii), $y = \frac{k}{x}$

Differentiating (ii) wr.t.x, we get $\frac{dy}{dx} = -\frac{k}{x^2}$ $\therefore$ Slope of tangent to (ii) at $(k^{2/3},\ k^{1/3}) = -\frac{k}{(k^{2/3})^2} = -\frac{1}{k^{1/3}}$ (iv)

The two curves cut at right angles (i.e., orthogonally) at $(k^{2/3},\ k^{1/3})$, if product of slopes of their tangents $= -1$

$\Rightarrow \left(\frac{1}{2k^{1/3}}\right)\left(-\frac{1}{k^{1/3}}\right) = -1 \Rightarrow 1 = 2k^{2/3} \Rightarrow 1 = 8k^2.$

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

SOLUTION

We have, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (i)

Differentiating (i) w.r.t. x, we get $\frac{2x}{a^2} - \frac{2y\frac{dy}{dx}}{b^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$ (ii) $\therefore$ Slope of tangent, to (i) at (x_0, y_0) is $\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = \frac{bx_0}{a^2y_0}$

Hence, the equation of the tangent to (i) at (x_0, y_0) is $y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0) \Rightarrow a^2y_0(y - y_0) = b^2x_0(x - x_0)$

$\Rightarrow b^2xx_0 - a^2yy_0 = b^2x_0^2 - a^2y_0^2 \Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$

[Dividing by a^2b^2] As (x_0, y_0) lies on (i), $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$

Hence the equation of the tangent is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1.$

Slope of the normal at $(x_0, y_0) = \frac{-1}{\text{Slope of tangent}} = -\frac{a^2y_0}{b^2x_0}$

$\therefore$ The equation of the normal at (x_0, y_0) $y - y_0 = -\frac{a^2y_0}{b^2x_0}(x - x_0) \Rightarrow \frac{y - y_0}{a^2y_0} = -\frac{(x - x_0)}{b^2x_0}$

$\Rightarrow \frac{y - y_0}{a^2y_0} + \frac{x - x_0}{b^2x_0} = 0.$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

SOLUTION

We have, $y = \sqrt{3x - 2}$ (i) and $4x - 2y + 5 = 0$ (ii) Slope of the line (ii) is 2

From (i), $\frac{dy}{dx} = \frac{3}{2\sqrt{3x - 2}}$

Let (x_1, y_1) be the point on (i) at which tangent is parallel to (ii), then $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ =Slope of line (ii)

$\Rightarrow \frac{3}{2\sqrt{3x_1 - 2}} = 2 \Rightarrow 3 = 4\sqrt{3x_1 - 2}$

$\Rightarrow 3x_1 - 2 = \left(\frac{3}{4}\right)^2 \Rightarrow x_1 = \frac{41}{48}$

Also, (x_1, y_1) lies in (i), therefore, $y_1 = \sqrt{3x_1 - 2} = \sqrt{3x\frac{41}{48} - 2.} = \sqrt{\frac{123 - 96}{48}} = \sqrt{\frac{27}{48}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$

$\therefore$ The point on (i) at which tangent is parallel to (ii) is $\left(\frac{41}{48}, \frac{3}{4}\right)$.

$\therefore$ Required equation of tangent is $y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$ or $48x - 24y - 23 = 0$

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is

(A) 3

(B) $\frac{1}{3}$

(C) -3

(D) $-\frac{1}{3}$

SOLUTION

(D) : We have, $y = 2x^2 + 3\sin x$ (i) $= \frac{dy}{dx} = 4x + 3\cos x$

Slope of the tangent to (i) at $x = 0$ is $\left(\frac{dy}{dx}\right)_{x=0} = 4 \cdot 0 + 3\cos 0 = 3$

So, slope of the normal to (i) at $x = 0$ is $\frac{-1}{\text{Slope of the tangent}} = -\frac{1}{3}$

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) $(1,\ 2)$

(B) $(2, 1)$

(C) $(1,\ -2)$

(D) $(-1, 2)$

SOLUTION

(A): We have, $y^2 - 4x = 0$ (i)

Slope of the line $y = x + 1$ is 1. From (i), $2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$

$\Rightarrow$ Slope of tangent to (i) is $\frac{2}{y}$

$\therefore \& \frac{2}{y} = 1 \Rightarrow y = 2$

When $y = 2$, then from (i)$2^2 = 4x \Rightarrow x = 1$ $\therefore$ The required point on the curve (i) is $(1,\ 2)$.