NCERT - Exercise 6.1

Evaluate the determinants in Exercises 1 and 2.

1. Find the rate of change of the area of a circle with respect to its radius r when

   (a) $r = 3$ cm

   (b) $r = 4$ cm

   SOLUTION

   Let $A = \pi r^2$ (1) (where A denotes the area of the circle when its radius is r) Differentiating (1), w.r.t. r, we get $= \dfrac{dA}{dr} = \pi(2r) = 2\pi r$

   (a) $\left(\dfrac{dA}{dr}\right)_{r=3\text{cm}} = 2\pi(3)\text{cm} = 6\pi\text{cm}^2/\text{cm}$

   (b) $\left(\dfrac{dA}{dr}\right)_{r=4\text{cm}} = 2\pi(4)cm = 8\pi\text{cm}^2/\text{cm}$

2. The volume of a cube is increasing at the rate of $8\text{cm}^3/\text{s}$. How fast is the surface area increasing when the length of an edge is 12cm?

   SOLUTION

   Let at any instant of time t, the edge of the cube be x, surface area be S and the volume be y then $V = x^3$ and $S = 6x^2$ (i)

   Differentiating (i) w.r.t. t, we get $\Rightarrow \dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$ (ii)

   and $\dfrac{dS}{dt} = 6(2x)\dfrac{dx}{dt}$ (iii)

   $\dfrac{dV}{dt} = 8\text{cm}^3/\text{sec}$ (Given)$\Rightarrow 3x^2\dfrac{dx}{dt} = 8\text{cm}^3/\text{sec}$ (using (ii))

   $\Rightarrow 3(12\text{cm})^2\dfrac{dx}{dt} = 8\text{cm}^3/\text{sec} \Rightarrow \dfrac{dx}{dt} = \dfrac{8}{432}\text{cm}/\text{sec} = \dfrac{1}{54}\text{cm}/\text{sec}$

   Substituting this value of $\dfrac{dx}{dt}$ in (iii), we get $\dfrac{dS}{dt} = 12(12\text{cm})\left(\dfrac{1}{54}\text{cm}/\text{sec}\right)$

   $= \dfrac{8}{3}\text{cm}^2/\text{sec} \therefore$ Rate of increase of surface area $= \dfrac{8}{3}\text{cm}^2/\text{sec}$.

3. The radius of a circle is increasing uniformly at the rate of 3cm/s. Find the rate at which the area of the circle is increasing when the radius is 10cm.

   SOLUTION

   Let at any instant of time t,

   the radius of the circle be r and its area be A. Then, $A = \pi r^2$

   $\Rightarrow \dfrac{dA}{dt} = \pi(2r)\dfrac{dr}{dt}$ and $\left(\dfrac{dA}{at}\right)_{r=l0\text{cm}} = 2\pi(10\text{cm})(3\text{cm/sec})$

   $= 60\pi\text{cm}^2/\text{sec}$

   $\therefore$ Rate of increase of area of the circle $= 60\pi\text{cm}^2/\text{sec}$.

4. An edge of a variable cube is increasing al the rate of 3cm/s. How fast is the volume of the cube increasing when the edge is 10cmlong?

   SOLUTION

   Let at any instant of time t, the edge of the cube be x and its volume be V then $V = x^3$ (i)

Differentiating (i) wrt. t, we get $\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3(10\text{cm})^2\,(3\text{cm/sec})$

$= 900\text{cm}^3/\text{sec}$ $\therefore$ Rate of increase of volume of the cube is $900\text{cm}^3/\text{sec}$.

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5cm/s. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area increasing?

   SOLUTION

   Let at any instant of time t, the radius of the circular wave be r and the area enclosed be A, then $A = \pi r^2$

   (i) Differentiating (i) w.r.t. t, we have $\Rightarrow \frac{dA}{dt} = \pi(2r)\frac{dr}{dt} = 2\pi(8cm)(5\text{cm/sec})$

   $= 80\pi\text{cm}^2/\text{sec}$

   $\therefore$ Rate of increase of enclosed area $= 80\pi\text{cm}^2/\text{sec}$.

6. The radius of a circle is increasing at the rate of 0.7cm/s . What is the rate of increase of its circumference?

   SOLUTION

   Let at any instant of time t, the radius of the circle be r and its circumference be C, then $C = 2\pi r$ (i)

   Differentiating (i) w.r.t. t, we get $\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi(0.7)\text{cm/sec} = (1.4\pi)\text{cm/sec}$

   Hence, the rate of increase of circumference $= (1.4\pi)\text{cm/sec}$

7. The length x of a rectangle is decreasing at the rate of 5cm/minute and the width y is increasing at the rate of 4cm/minute. When $x = 8\text{cm}$ and $y = 6\text{cm}$, find the rates of change of

   (a) perimeter, and

   (b) the area of the rectangle.

   SOLUTION

   Let at any instant of time t, length of rectangle be x, breadth be y, the perimeter P and the area be A, then

   (a) We have, $\frac{dx}{dt} = -5\text{cm/min}$ and $\frac{dy}{dt} = 4\text{cm/min}$ (a) $P = 2(x+y)$ (i)

   Differentiating (i) w.r.t. t, we get $\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4)\text{cm/min} = -2\text{cm/min}$

   $\therefore$ Perimeter of the rectangle is decreasing at the rate of 2cm/min

   (b) $A = xy$ (ii) Differentiating (i) w.r.t. t, we get $\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = (8\text{cm})\,(4\text{cm/min}) + (6\text{cm})\,(-5\text{cm/min}) = 2\text{cm}^2/\text{min}$ $\therefore$ Area of the rectangle is increasing at the rate of $2\text{cm}^2/\text{min}$.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

   SOLUTION

   Let at any instant of time t, the radius of the balloon be r and its volume be V, then $V = \frac{4}{3}\pi r^3$ (i)

   Differentiating (i) w.r.t. t, we get $\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right)$

   $\Rightarrow 900\text{cm}^3/\text{sec} = \left(\frac{4}{3}\pi\right)\left\{3(15cm)^2\frac{dr}{dt}\right\}$

   $\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi \times (15)^2}\text{cm/sec} = \frac{1}{\pi}\text{cm/sec}$

   $\therefore$ Rate of increase of the radius of the balloon $= \frac{1}{\pi}\text{cm/sec}$.

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

SOLUTION

Let at any instant of time, the radius of the balloon be r and its volume be V, then $V = \frac{4}{3}\pi r^3$ (i)

Differentiating (i) w.r.t. r, we get $\frac{dV}{dt} = \left(\frac{4}{3}\pi\right) 3r^2 = 4\pi r^2 = 4\pi(10\text{cm})^2 = 400\pi\text{cm}^3$

Rate of increase of volume with respect to change in radius $= 400\pi\text{cm}^3/\text{cm}$

10. A ladder 5m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4m away from the wall ?

SOLUTION

If the foot of the ladder is at a distance x from the wall and the top is at a vertical height of y at any instant of time t, then $(5)^2 = x^2 + y^2$ (i)

Differentiating (i) w.r.t. t, we have $\Rightarrow \frac{d}{dt}(25) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ (ii)

We have $\frac{dx}{dt} = 0.02\text{m/sec}$ $x = 4\text{m}$ and $y = \sqrt{5-4^2}\text{m} = 3\text{m}$

Hence, from (ii) $0 = 2 \times 4\text{m} \times 0.02\text{m/sec} + 2 \times 3\text{m}\frac{dy}{dt}$

$\Rightarrow \frac{dy}{dt} = -\frac{0.16}{6}\text{m/sec}$

$\therefore$ Rate of decrease of height of the ladder on the wall $= \frac{16}{600}\text{m/sec} = \frac{1600}{600}\text{cm/sec} = \frac{8}{3}\text{cm/sec}$

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the $y-$ coordinate is changing 8 times as the $x-$coordinate.

SOLUTION

We have the curve,$6y = x^3 + 2$ ...(i) and $\frac{dy}{dt} = 8\frac{dx}{dt}$

Differentiating (i) w.r.t. t, we obtain $6\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 6 \times \frac{8dx}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt}\left(48 - 3x^2\right) = 0$

$\frac{dx}{dt}$ cannot equal to zero $\therefore 48 - 3x^2 = 0 \Rightarrow x = \pm 4$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

When $x = 4 \Rightarrow 6y = 4^3 + 2 \Rightarrow y = \frac{66}{6} = 11$

When $x = -4 \Rightarrow 6y = (-4)^3 + 2 \Rightarrow y = \frac{-62}{6} = \frac{-31}{3}$

Hence, the required points are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$.

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$cm/s. At what rate is the volume of the bubble increasing when the radius is 1cm?

SOLUTION

Let at any instant of time t, the radius of the bubble be r and its volume be V, then $V = \frac{4}{3}\pi r^3$ (i)

Differentiating (i), w.r.t. t, we get $\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$

$= 4\pi(1\text{cm})^2\left(\frac{1}{2}\text{cm/sec}\right) = 2\pi\text{cm}^3/\text{sec}$

Hence, the rate of increase of the volume of the bubble= $2\pi \text{cm}^3/\text{sec}$.

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

SOLUTION

Diameter of the balloon, $d = \frac{3}{2}(2x+1)$ $\therefore$ Radius of the balloon, $r = \frac{d}{2} = \frac{1}{2}\left\{\frac{3}{2}(2x+1)\right\} = \frac{3}{4}(2x+1)$

So, the volume V of the balloon $V = \frac{4}{3}\pi(\text{radius})^3 = \frac{4}{3}\pi\left\{\frac{3}{4}(2x+1)\right\}^3 = \frac{9\pi}{16}(2x+1)^3$ (i)

Differentiating (i) w.r.t. x, we get $\frac{dV}{dx} = \frac{9\pi}{16} \times 3(2x+1)^2 \times 2 = \frac{27\pi}{8}(2x+1)^2$

14. Sand is pouring from a pipe at the rate of $12\text{cm}^3/\text{s}$.

The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4cm?

SOLUTION

Let at any instant of time t, the radius of the base of the cone be r, its height be h and the volume of the sand cone be V, then $h = \frac{1}{6}r \Rightarrow r = 6h$ and $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$ (i)

Differentiating (i) w.r.t. t, we get $\frac{dV}{dt} = (12\pi)\left(3h^2\frac{dh}{dt}\right)$

$\Rightarrow 12\text{cm}^3/\text{sec} = 36\pi(4\text{cm})^2\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi \times 16}\text{cm}/\text{sec} = \frac{1}{48\pi}\text{cm}/\text{sec}$

$\therefore$ Rate of increase of the height of the sand cone $= \frac{1}{48\pi}$ cm/sec.

15. The total cost $C(x)$ in rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

SOLUTION

We have, $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$ (i)

Differentiating (i) w.r.t. x, we get

Marginal cost $= \frac{dC}{dx} = 0.007 \times 3x^2 - 0.003 \times 2x + 15 + 0$

$\therefore$ Marginal cost when 17 units are produced $= Rs20.967$.

16. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 2x + 15$. Find the marginal revenue when $x = 7$.

SOLUTION

We have, $R(x) = 13x^2 + 26x + 15$.(i) Differentiating (i) wr.t. x, we have.

Marginal revenue $= \frac{dR}{dx} = 13 \times 2x + 26 = 26x + 26$ $\therefore$ $\left(\frac{dR}{dx}\right)_{x=7} = 26 \times 7 + 26 = 208 \Rightarrow$ Marginal revenue (when $x = 7$)$= Rs208$
.

Choose the correct answer in the Exercises 17 and 18.

17. The rate of change Of the area of a circle with respect to its radius r at $r = 6$ cm is (A) 10π

(B) 12π

(C) 8π

(D) 11π

SOLUTION

(B) If A is the area of the circle corresponding to radius r, then $A = \pi r^2$ (i) Differentiating (i) w.r.t. r, we get $\frac{dA}{dr} = 2\pi r$ $\therefore$ $\left(\frac{dA}{dr}\right)_{r=6\text{cm}} = 2\pi(6\text{cm}) = 12\pi\text{cm}$

18. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

(A) 116

(B) 96

(C) 90

(D) 126

SOLUTION

(D) We have, $R(x) = 3x^2 + 36x + 5$ (i)

Differentiating (i) w.r.t. x, we get, marginal revenue $\frac{dR}{dx} = \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36 = \left(\frac{dR}{dx}\right)_{x=15} = 6 \times 15 + 36 = 126$