NCERT - Miscellaneous Exercise

1. Let f: R−R be defined as f(x)= 10x + 7. Find the function g : R → R such that gof = fog = IR.

SOLUTION

f: X →Y, where X,$Y \subseteq R$. Let y ∈Y , arbitrarily.

By definition, y = 10x + 7 for x ∈ X

$\Rightarrow x = \frac{y-7}{10}$

We define, g : Y → X by g(y) $=\frac{y-7}{10}$

Now, (gof)(x) = $g(f(x)) = \frac{f(x)-7}{10} = \frac{(10x+7)-7}{10} = x$

and (fog) (y) = f (g(y)) = 10g (y) + 7 = 10 $\left(\frac{y-7}{10}\right)$ + 7= y

Thus, gof = fog = IR.

Hence, f is invertible and g : Y → X such that g(y) = $\frac{y-7}{10}$

2. Let f: W→W be defined as f(n) =n−1, if n is odd and f (n) = n + 1, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

SOLUTION

f : W→ W

$f(n) = \begin{cases} n-1, & if \quad \text{nisodd} \\ n+1, & if \quad \text{niseven} \end{cases}$

Injectivity

Let n, m be any two odd real whole numbers.

$\therefore f(n) = f(m) \Rightarrow n-1 = m-1 \Rightarrow n = m$

Again, let n, m be any two even whole numbers.

$\therefore f(n) = f(m) \Rightarrow n+1 = m+1 \Rightarrow n = m$ If n is even and m is odd, then n ≠ m.

Also, if f(n) is odd and f(m) is even, then f(n)≠f(m)

Thus, if $n \neq m \Rightarrow f(n) \neq f(m)$ ∴ f is an injective.

Surjectivity :

Let n be an arbitrary whole number.

If n is an odd number, then there exists an even whole number

(n + 1) such that f(n + 1) =n + 1−1 =n

If n is an even number, then there exists an odd whole number,

such that f(n−1)=(n −1) + 1 =n

Thus, every $n \in W$ has its pre-image in W.

So, f: W→W is a surjective.

Thus, f is invertible and f^{-1} exists.

Now, f(n−1) = n, if n is odd and f(n + 1) =n, if n is even.

⇒ n−1 =f^{-1}(n), if n is odd and n + 1 = $f^{-1}(n)$, if n is even.

Hence, $f^{-1}(n) = \begin{cases} n-1, & if \quad \text{nisodd} \\ n+1, & if \quad \text{niseven} \end{cases}$. Hence, $f^{-1} = f$

3. If f : R → R is defined by f(x) = $x^2 - 3x + 2$, find f(f(x)).

SOLUTION

We are given that, f(x) = $x^2 - 3x + 2$

$\therefore f[f(x)] = f(x^2 - 3x + 2)$ /[/«] = /(x^2– 3* + 2)

$\Rightarrow f[f(x)] = (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

$= x^4 - 6x^3 + 10x^2 - 3x.$

Hence, f(f(x)) = $x^4 - 6x^3 + 10x^2 - 3x.$

4. Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by f(x) = $\frac{x}{1+|x|}, x \in R$is one-one and onto function.

SOLUTION

We have : f (x)= $\frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x}, & if \quad x \geq 0 \\ \frac{x}{1-x}, & if \quad x < 0 \end{cases}$

Here, Domain of f = R

To prove : f is one−one Let x, y ∈ Domain of f=R, such that x ≠ y Here, four cases arise.

Case I : When $x \geq 0, y \geq 0$

If $x \neq y \Rightarrow 1 + x \neq 1 + y \Rightarrow \frac{1}{1+x} \neq \frac{1}{1+y} \Rightarrow \frac{1}{1+x} \neq \frac{-1}{1+y}$

$\Rightarrow 1 - \frac{1}{1+x} \neq 1 - \frac{1}{1+y} \Rightarrow \frac{x}{1+x} \neq \frac{y}{1+y}$

$\Rightarrow f(x) \neq f(y).$

Case II : When x ≥ 0 and y < 0 Then, f(x) = $\frac{x}{1+x} \geq 0$ and $f(y) = \frac{y}{1-y} < 0$

$\Rightarrow f(x) \neq f(y).$

Case III : When $x < 0$ and y ≥ 0

Then, $f(x) < 0$ and f (y)≥ 0 [As in Case II]

$\Rightarrow f(x) \neq f(y)$

Case IV : When x ≤0 and y ≤ 0

If $x \neq y \Rightarrow -x \neq -y \Rightarrow 1 - x \neq 1 - y \Rightarrow \frac{1}{1-x} \neq \frac{1}{1-y}$

$\Rightarrow \frac{1}{1-x} - 1 \neq \frac{1}{1-y} - 1 \Rightarrow \frac{x}{1-x} \neq \frac{y}{1-y} \Rightarrow f(x) \neq f(y).$

Thus, in each case, $x \neq y \Rightarrow f(x) \neq f(y).$

Hence, f is one−one.

To prove : f is onto

Let y ∈ R, where y is arbitrary.

Then,$y = f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} < 1, & if \quad x \geq 0 \\ \frac{x}{1-x} > -1, & if \quad x \leq 0 \end{cases}$

Case I When y=$\frac{x}{1+x}$, where y ≥ 0

y + xy = x or y = x (1−y) or x =$\frac{y}{1-y}$ ≥0

Case II When y $=\frac{x}{1+x}$, where $y<0$

y $-$xy $=$ x or y $=$ x + xy or x $=\frac{y}{1+y}<0$

Thus, when y ≥ 0, there is $\frac{y}{1-y}\in$ Domain of f $=$ R such that

$$f\left(\frac{y}{1-y}\right)=\frac{\frac{y}{1-y}}{1+\frac{y}{1-y}}=\frac{y}{1-y+y}=\frac{y}{1}=y$$

and when $y<0$, there is $\frac{y}{1+y}\in$ Domain of f=R such that $f\left(\frac{y}{1+y}\right)=\frac{\frac{y}{1+y}}{1-\frac{y}{1+y}}=\frac{y}{1+y-y}$

$=\frac{y}{1}=y$

Hence, f is onto.

5. Show that the function f : R $\rightarrow$ R given by f(x) $=$ x3 is injective.

SOLUTION

Let $x_1,x_2\in R$ be such that,

$f(x_1)=f(x_2)\Rightarrow x_1^3=x_2^3\Rightarrow x_1=x_2$

$\therefore$ f is one$-$one. Hence, f(x) $=x^3$ is injective.

6. Give example of two functions f : N $\rightarrow$ Z and g : Z $\rightarrow$ Z such that gof is injective but g is not injective.

(Hint : Consider f(x) $=$ x and g(x) $=|x|$)

SOLUTION

f : N $\rightarrow$ N and g : Z $\rightarrow$ Z

Let f(x) $=$ x and g(x) =|x|. Since, g(x)=g($-$x) $=|x|\forall x\in Z$

$\therefore$ g is not one$-$one $\Rightarrow$ g is not injective.

Since, f : N $\rightarrow$ Z and g : Z $\rightarrow$ Z $\Rightarrow$ gof : N $\rightarrow$ Z. Let $x_1,x_2\in N$.

Now, (gof) $(x_1)=(gof)(x_2)\Rightarrow g(x_1)=g(x_2)|x_1|=|x_2|$

$\Rightarrow x_1=x_2$ [0]

$\therefore$ gof is one-one. Hence, gof is injective.

7. Give example of two functions f : N $\rightarrow$ N and g : N $\rightarrow$ N such that gof is onto but f is not onto.

(Hint : Consider f(x) $=$ x + 1 and g(x)=$\begin{cases} x-1, & if \quad x>1 \\ 1, & if \quad x=1 \end{cases}$

SOLUTION

Consider, f(x)=x + 1 and g(x)=$\begin{cases} x-1, & if \quad x>1 \\ 1, & if \quad x=1 \end{cases}$

$f(x)=x+1\geq 1+1\forall x\in N\Rightarrow f(x)\geq 2\forall x\in N.$

Clearly, range of $f\neq N$ [1$\notin$Range of f]

$\therefore$ f is not onto.

Now, (gof) : N $\rightarrow$ N such that (gof) (x)=g(f(x))=g(x + 1) $=$ (x + 1)$-$1 [x + 1 $>$ 1 for all x $\in$N] $=x\forall x\in N$

$\therefore$Range of (gof) $=$ N [gof is identity function] Hence, gof is onto.

8. Given a non empty set X, consider P(X) which is the set of all subsets of X. Define the relation R in P(X) as follows : For subsets A, B in P(X), ARB if and only if A$\subset$B. Is R an equivalence relations on P(X) ? Justify your answer.

SOLUTION

(i) Since A $\subset$A $\forall A \in$ P (X) $\Rightarrow$ ARA

$\therefore$ R is reflexive.

(ii) Let ARB $\Rightarrow$ A $\subset$B and BRA $\Rightarrow$ B $\subset$ A

$\Rightarrow$ A = B (which is not so) [$\Rightarrow$ ARB $\not\Rightarrow$BRA $\Rightarrow$ R is not symmetric

(iii) ARB,BRC $\Rightarrow$ A$\subset$B, B$\subset$ C $\Rightarrow$ f $\subset$C $\Rightarrow$ ARC $\Rightarrow$ R is transitive

Thus R is not an equivalence relation of P(X).

9. Given a non-empty set X, consider the binary operation $*$: P(X) × P(X) → P(X) given by A$*$ B = A$\cap$B $\forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

SOLUTION

(i) Let E $\in$ P(X) be the identity element.

Then, A $*$ E = E $*$ A = A $\forall$A $\in$ P(X)

$\Rightarrow$ A $\cap$ E = E $\cap$ A = A $\forall$A $\in$ P(X)

$\Rightarrow$ X $\cap$ E = X because X $\in$P(X) $\Rightarrow$ X $\subset$E.

Also, Thus, E = X. Hence, X is the identity element.

(ii) Let A $\in$ P(X) be invertible. Then, there exists B $\in$ P(X such that A $*$ B = B $*$ A = X, where X is the identity element.

$\Rightarrow$ A $\cap$ B = B $\cap$ A = X $\Rightarrow$ X $\subset$ A, X $\subset$ B Also, A, B

$\therefore$ A = X = B.

Hence, X is the only invertible element and A^{-1}= B = X.

10. Find the number of all onto functions from the set $\{1,2,3,....,n\}$ to itself.

SOLUTION

The number of onto functions that can be defined from a finite set X containing n elements on to a finite set Y containing n elements.

Let X : $\{1,2,....,n\}$ and Y : $\{1,2,3,....,n\}$

One of the elements of set X(say 1) has any one of the pre-image 1, 2, ..., n i.e. n ways.

In similar way, the element (say 2) in (n−1) ways

$\therefore$ Total number of possible ways=n (n−1) (n −2) 3.2.1

= n!

11. 11. Let S = $\{a,b,c\}$ and T = $\{1,2,3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = $\{(a,3),(b,2),(c,1)\}$

(ii) F= $\{(a,2),(b,1),(c,1)\}$

SOLUTION

Given, S = $\{a,b,c\}$ and T = $\{1,2,3\}$. (i) F = $\{(a,3),(6,2),(c,1)\}$

i.e. F(a) = 3, F(b) = 2, F(c) = 1

$\Rightarrow F^{-1}(3)$ = a, $F^{-1}(2\ )$ = b, $F^{-1}(1)$=c

$\therefore F^{-1} = \{(3,a),(2,b),(1,c)\}$

. (ii) F = $\{(a,2),(b,1)(c,1)\}$

F is not one-one function, since element b and c have the same image 1, so F^{-1} does not exist.

12. Consider the binary operations $*$: R × R → R and o : R × R→ R defined as a $*$ b = $|a - b|$ and aob = a, $\forall a,b \in R$. Show that $*$ is commutative but not associative and o is associative but not commutative. Further, show that $\forall$a, b, c $\in$R, a $*$ (boc) = (a $*$ b) o (a $*$ c). (If it is so, we sav that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

SOLUTION

For commutativity :

a $*$ b = $|a - b|$ and b $*$ a = $|b - a| = |a - b|$

$\therefore$ a $*$ b = b $*$ a

Thus, the operation $*$ is commutative.

For associativity

Consider, a $*$ (b $*$ c) = a $*|b - c| = |a - |b - c||$

Also (a $*$ b) $*c = |a - b| * c = ||a - b| - c|$

$\therefore$ a $*$ (b $*$ c) $\neq$(a $*$ b) $*$ c

Thus, the operation $*$ is not associative.

For commutativity

aob = a $\forall$a, b $\in$ R. Now, boa = b $\Rightarrow$ aob $\neq$ boa

Thus, operation o is not commutative.

For associativity

ao (boc) = aob = a and (aob) oc = aoc = a

ao (boc) = (aob) oc. Thus, operation o is associative.

To prove : a $*$ (boc) = (a $*$ b) o (a $*$ c)

L.H.S. = $a * (boc) = a * b = |a - b|$

R.H.S. = $(a * b)o(a * c) = |a - b|o|a - c| = |a - b|$

Thus, a $*$ (boc) = (a $*$ b) o (a $*$ c). Hence proved.

Another distributive law

ao(b $*$ c) = (aob) $*$ (aoc)

L.H.S. = $ao(b * c) = ao(|b - c|) = a$

R.H.S. = $(aob) * (aoc) = a * a = |a - a| = 0.$

As, L.H.S.$*$ R.H.S. Hence, the operation o does not distribute over $*$.

13. Given a non-empty set X, let$*$: P(X) × P(X) → P(X) be defined as A $*$ B = (A$-B$)$\cup$(B - A), $\forall$ A, B $\in$ P(X). Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of P(X)are invertible with A^{-1} = A.

(Hint : (A $-\phi$) $\cup$($\phi-$A) = A and (A$-$A)$\cup$(A $-$A) = A $*$ A = ϕ).

SOLUTION

To show : ϕ is the identity

For every A $\in P(X)$, we have

$\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$

and $A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$

$\Rightarrow$ ϕ is the identity element for the operation $*$ on P(X).

Also, A$*$A = (A$-$A) $\cup$ (A$-$A) = $\phi \cup \phi = \phi$

$\Rightarrow$ Every element A of P(X) is invertible with $A^{-1} = A$

14. Define a binary operation $*$ on the set 0, 1, 2, 3, 4, 5 as $a*b = \begin{cases} a+b, & if \quad a+b<6 \\ a+b-6, & if \quad a+b \geq 6 \end{cases}$

Show that zero is the identity forth is operation and each element$a \neq 0$ of the set is invertible with $6-a$ being the inverse of a.

SOLUTION

For identity

If e be the identity element, then a $*$ e = e $*$ a = a

Now, a $*$ 0 = a + 0 = a and 0 $*$ a = 0 + a = a

Thus, a $*$ 0 = 0 $*$ a = a. Hence, 0 is the identity element of the operation.

For inverse

If b be the inverse of a. then a $*$ b = b $*$ a = e.

Now a $*$ (6$-$ a) = a + (6 $-$ a) $-$ 6 = 0

and (6 $-$ a) $*$ a = (6 $-$ a) + a $-$ 6 = 0.

Hence, each element a of the set is invertible with inverse 6$-$a.

15. Let A= { - 1,0,1,2}, B= { - 4, - 2,0,2} and f, g : A → B be function defined by f(x)= $x^2 - x, x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1, x \in A$.
Are f and g equal ? Justify your answer.
(Hint : One may note that two functions f : A→B and g : A→B such that f(a)=g(a) $\forall a \in A$, are called equal functions).

SOLUTION

When x $=-1$, f(-1) = 12 + 1 = 2

and $g(-1) = 2\left|-1 - \frac{1}{2}\right| - 1 = 2$

When x = 0, f(0) = 0 and g(0) $=2\left|-\frac{1}{2}\right| - 1 = 2 \times \frac{1}{2} - 1 = 0$

When x =1, f(1)=$1^2 - 1$=0

and g(1) $= 2\left|1 - \frac{1}{2}\right| - 1 = 2 \times \frac{1}{2} - 1 = 0$

When x = 2, f(2)=22−2=2 and g(2) $=2\left|2 - \frac{1}{2}\right| - 1 = 3 - 1 = 2$

Thus, for each a ∈ A, f(a) = g(a). Hence, f and g are equal functions.

16. Let A = $\{1,2,3\}$. Then number of relations containing (1, 2) and (1, 3), which are reflexive and symmetric but not transitive is
(A) 1
(B) 2
(C) 3
(D) 4

SOLUTION

(A) There is only one relation containing (1, 2) and (1, 3) which is reflexive and symmetric but not transitive.

17. Let A = $\{1,2,3\}$. Then number of equivalence relations containing (1, 2) is
(A) 1
(B) 2
(C) 3
(D) 4

SOLUTION

(B) : There are two equivalence relations containing (1, 2).

18. Let f : R → R be the Signum Function defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ and g : R→R be the Greatest Integer Function given by g(x)=[x], where [x] is greatest integer less than or equal to x. Then, does fog and gof coincide in (0, 1] ?

SOLUTION

For x ∈ (0, 1]

(fog) (x) $=f(g(x))$=$f([x])$

$$= \begin{cases} f(0), & if \quad 0 < x < 1 \\ f(1), & if \quad x = 1 \end{cases} = \begin{cases} 0, & if \quad 0 < x < 1 \\ 1, & if \quad x = 1 \end{cases} \quad \ldots(1)$$

And (gof) (x) = g(f(x)) =g(1) = [1] = 1

$\Rightarrow (gof)(x) = 1 \forall x \in$ (0, 1] ...(2)

From (1) and (2), (fog) and (gof) do not coincide in (0, 1].

19. Number of binary operations on the set $\{a,b\}$ are

(A) 10

(B) 16

(C) 20

(D) 8

SOLUTION

(B) There are two elements in the set $\{a,b\}$. $\therefore$ Number of binary operations $= 24 = 16$.