NCERT - Exercise - 1.4

1. Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.

   (i) Or Z+, define $*$ by a$*$ b= a−b

   (ii) On Z+, define $*$ by a$*$ b = ab

   (iii) On R, define $*$ by a $*$ b = ab^2

   (iv) On Z+, define $*$ by a $*$ b = $|a-b|$

   (v) On Z+, define $*$ by a $*$ b = a

   SOLUTION

   (i) Z+ = $\{1,2,3,...\}$, we have a $*$ b = a−b

   Let a = 1, b = 3 ⇒ a$*$ 6 = 1−3 =−2 ∉Z+

   Hence, the operation $*$ is not a binary operation on Z+.

   (ii) Z+= $\{1,2,3,\}$, we have a $*$ b = ab

   Let a = 2, 6 = 4 ⇒ a$*$b = 2 $*$ 4 = 8 ∈Z+

   Hence, the operation $*$ is a binary operation on Z+.

   (iii) R (set of real numbers), we have a $*$ b = ab^2

   Let a = 5.2, b = 3 ⇒ a$*$b = 5.2(3)2 = 46.8 ∈R

   Hence, the operation $*$ is a binary operation on R.

   (iv) Z+ = $\{1,2,3,....\}$, we have a $*$ b = $|a-b|$

   Let a =3, b = 7 ⇒ a $*$ b = |3−7| = $|-7|$ = 4 ∈ Z^+ Hence, the operation $*$ is a binary operation on Z+.

   (v) Z+ = $\{1,2,3,....\}$, we have a $*$ b = a

   Let a=5, b=7 ⇒ a$*$b = 5 ∈ Z^+

   Hence, the operation $*$ is a binary operation on Z^+.

2. For each operation $*$defined below, determine whether $*$ is binary, commutative or associative.

   (i) On Z, define a $*$ b = a− b

   (ii) On Q, define a$*$b= ab + 1

   (iii) On Q, define a $*$ b = $\frac{ab}{2}$

   (iv) On Z+, define a$*$ b= 2ab

   (v) On Z+, define a$*$b= ab

   (vi) On R−$\{-1\}$, define a $*$ b=$\frac{a}{b+1}$

   SOLUTION

   (i) a $*$ b = a−b on Z

   For commutativity

   a $*$ b = a − b and b $*$ a = b − a =− (a − b) ≠a $*$ b

   ⇒ a $*$ b ≠ b $*$ a

   For associativity

   a $*$ (b $*$ c) = a $*$ (b − c) = a − (b − c) = (a − b + c)

   And (a $*$ b) $*$ c = (a − b) $*$ c= a − b− c.

   ∴ a $*$ (b $*$ c) ≠ (a $*$ b) $*$ c

   Thus, the operation $*$ is neither commutative nor associative.

(ii) a ∗ b = ab + 1 on Q

For commutativity

a ∗ b = ab + 1 and b ∗ a = ba + 1 = ab + 1. a ∗ b = b ∗ a

For associativity

a ∗ (b ∗ c) = a ∗ (bc + 1) = a (bc + 1) + 1 = abc + a + 1

And, (a ∗ b) ∗ c = (ab + 1) ∗ c = (ab + 1) c +1 = abc + c + 1

∴ a ∗ (b ∗ c) ≠ (a ∗ b) ∗ c

Thus, the operation ∗ is commutative but not associative.

(iii) a∗ b= $\frac{ab}{2}$on Q

For commutativity

a∗ b = $\frac{ab}{2}$ and b∗ a = $\frac{ba}{2} = \frac{ab}{2}$

∴ a ∗ b = b ∗ a

For associativity

a∗ (b∗ c) = a∗$\left(\frac{bc}{2}\right) = \frac{abc/2}{2} = \frac{abc}{4}$

and, (a∗ b)∗ c = $\left(\frac{ab}{2}\right)$ ∗(c) = $\left(\frac{abc/2}{2}\right) = \frac{abc}{4}$

∴ a ∗ (b ∗ c) = (a ∗ b) ∗ c

Thus, the operation ∗ is commutative and also associative,

(iv) a ∗ b = 2ab, on Z+

For commutativity

a ∗ b =2ab and b ∗ a = 2ba = 2ab. ∴ a ∗ b = b ∗ a

For associativity

a∗ (b∗ c)=a∗(2bc)=(2)$^{a\cdot 2^{bc}}$ and (a∗ b)∗ c = (2ab) ∗ c= $2^{2^{ab}\times c}$

a ∗ (b ∗ c) ≠ (a ∗ b) ∗ c

Hence, the operation ∗ is commutative but not associative.

(v) a ∗ b = ah on Z^-

For commutativity

a ∗ b = ah and b ∗ a = ba. ∴ a ∗ b ≠ b ∗ a

For associativity

a∗ (b ∗ c)= a∗ (bc)= (a)$^{b^c}$ and (a ∗ b) ∗ c= (ab) ∗ c = (ab)c= abc

Thus, a ∗ (b ∗ c) ≠ (a ∗ b) ∗ c

Hence, the operation ∗ is neither commutative nor associative.

(vi) a ∗ b = $\frac{a}{b+1}$ on R −{ - 1}

For commutativity

a∗ b = $\frac{a}{a+1}$ and b∗ a = $\frac{b}{a+1}$ $\therefore a*$ b $\neq b*a$

For associativity

$$a*(b*c) = a*\left(\frac{b}{c+1}\right) = \frac{a}{\frac{b}{c+1}+1} = \frac{a(c+1)}{b+c+1}$$

and $(a*b)*c = \left(\dfrac{a}{b+1}\right)*c = \dfrac{\frac{a}{b+1}}{c+1} = \dfrac{a}{(b+1)(c+1)}$

$\therefore$ a $*$ (b $*$ c) $\neq$ ($a*$ b) $*$ c

Hence, the operation $*$ is neither commutative nor associative.

3. Consider the binary operation $()$ on the set $\{1,2,3,4,5\}$ defined by $a^b = min\{a,b\}$. Write the operation table of the operation A.

SOLUTION

Let A = $\{1,2,3,4,5\}$

a^b = minimum of a and b

4. Consider a binary operation $*$ on the set $\{1,2,3,4,5\}$ given by the following multiplication table.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

(i) Compute (2 $*$ 3) $*$ 4 and 2 $*$ (3 $*$ 4).

(ii) Is $*$ commutative ?

(iii) Compute (2 $*$ 3) $*$ (4 $*$ 5).

(Hint : use the following table)

SOLUTION

(i) 2 $*$ 3 = 1 and 3 $*$ 4 = 1

Now, (2 $*$ 3) $*$ 4 = 1 $*$ 4 = 1 and 2 $*$ (3 $*$ 4) = 2 $*$ 1 = 1

(ii) 2 $*$ 3 = 1 and 3 $*$ 2 = 1 $\therefore$ 2 $*$ 3 = 3 $*$ 2

Hence, the operation is commutative.

(iii) (2 $*$ 3) $*$ (4 $*$ 5) = 1 $*$ 1 = 1.

5. Let $*$ be binary operation on the set $\{1,2,3,4,5\}$ defined by a $*$' b = H.C.F. of a and b. Is the operation $*$' same as the operation $*$ defined in Q. 4 above? Justify your answer.

SOLUTION

Let A = $\{1,2,3,4,5\}$

a$*$' b = HCF of a and b is given by

We observe that the operation $*$' is the same as the operation $*$ in Q. 4.

6. Let $*$ be the binary operation on N given by a$*$ b= L.C.M. of a and b. Find

(i) 5 $*$ 7, 20 $*$ 16

(ii) Is $*$ commutative ?

(iii) Is $*$ associative ?

(iv) Find the identity of $*$ in N

(v) Which elements of N are invertible for the operation $*$?

SOLUTION

a $*$ b = L.C.M. of a and b.

(i) 5 $*$ 7 = L.C.M. of 5 and 7 = 35

20 $*$ 16 = L.C.M. of 20 and 16 = 80

(ii) a $*$ b = L.C.M. of a and b = L.C.M. of b and a = b $*$ a.

Thus, operation $*$ is commutative.

(iii) a $*$ (b $*$ c) = a $*$ (L.C.M. of b and c)

= L.C.M. of (a and (L.C.M. of b and c))

= L.C.M. of a, b and c.

Similarly, (a $*$ b) $*$ c = (L.C.M. of a and b) $*$ c

= L.C.M. of ((L.C.M. of a and b) and c) = L.C.M. of a, b and c

Thus, a $*$ (b $*$ c) = (a $*$ b) $*$ c

Hence, the operation $*$ is associative.

(iv) Identity of $*$ in N = 1 because, a $*$ 1

= L.C.M. of a and 1 = 1

(v) Only the element 1 in N is invertible for the operation $*$ because $1 * \frac{1}{1} = 1$.

7. Is $*$ defined on the set $\{1,2,3,4,5\}$ by a $*$ b=L.C.M. of a and b a binary operation? Justify your answer.

SOLUTION

Let A = $\{1,2,3,4,5\}$ and a $*$ b = L.C.M. of a and b.

2 $*$ 3 = L.C.M. of 2 and 3 = 6 $\notin$A

Hence, the operation $*$ is not a binary operation.

8. Let $*$ be the binary operation on N defined by a $*$ b = H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N ?

SOLUTION

Commutativity

a$*$ b = H.C.F. of a and b = H.C.F. of b and a = b $*$ a

Thus, operation $*$ is commutative.

Associativity

(a $*$ b) $*$ c = (H.C.F. of a and b) $*$ c

= H.C.F. of [(H.C.F. of a and b) and c] = H.C.F. a, b and c

a $*$ (b $*$ c) = a $*$ [H.C.F. of b and c]

= H.C.F. of [a and (H.C.F. of b and c)] = H.C.F. of [a, b and c]

$\Rightarrow$ (a $*$ b) $*$ c = a $*$ (b $*$ c). Thus, operation $*$ is associative.

Identity

Now, 1 $*$ a = a $*$ 1 $\neq$ a

There does not exist any identity element.

9. Let $*$ be a binary operation on the set Q of rational numbers as follows:

(i) a $*$ b = a$-$b

(ii) a $*$ b = $a^2 + b^2$

(iii) a $*$ b = a + ab

(iv) a $*$ b = (a$-b)^2$

(v) a $*$ b = $\frac{ab}{4}$

(vi) a $*$ b = ab^2

Find which of the binary operations are commutative and which are associative.

SOLUTION

(i) For commutativity :

a $*$ b = a$-$b =$-$(b $-$ a) =$-$ b $*$ a

Thus, the operation $*$ is not commutative.

For associativity :

(a∗b)∗c=(a−b)∗c=(a−b)−c=a−b−c

And, a∗(b∗c)=a∗(b−c)=a−(b−c)=a−b + c ∴ (a∗b)∗c≠a∗(b∗c)

Thus, the operation ∗ is not associative.

(ii) For commutativity

a ∗ b $-a^2+b^2 = b^2+a^2 =$ b ∗ a Thus, the operation ∗ is commutative.

For associativity

(a ∗ b) ∗ c = $(a^2+b^2)*c = (a^2+b^2)+c^2$

and a ∗ (b ∗ c) = a $*(b^2+c^2) = a^2+(b^2+c^2)$

Thus, (a ∗ b) ∗ c ≠ a ∗ (b ∗ c)

Hence, the operation ∗ is not associative.

(iii) For commutativity

a ∗ b = a + ab = a(1 + b) and b ∗ a = b + ba = b(1 + a)

∴ a ∗ b ≠ b ∗ a

Thus, the operation ∗ is not commutative. For associativity

(a ∗ b) ∗ c = (a + ab) ∗ c = (a + ab) + (a + ab)c

and a ∗ (b ∗ c) = a ∗ (b + bc) = a + a(b + bc)

∴ (a ∗ b) ∗ c ≠ a ∗ (b ∗ c)

Hence, the operation ∗ is not associative.

(iv) For commutativity a ∗ b = (a $-b)^2 =$ (b $-a)^2 =$ b ∗ a.

Thus, the operation ∗ is commutative.

For associativity

a ∗ (b ∗ c) = a ∗ (b$-c)^2 =$ [a− (b $-c)^2]^2$

and (a ∗ b) ∗ c = $(a-b)^2 * c = [(a - b)^2 - c]^2$

∴ a ∗ (b ∗ c) ≠ (a ∗ b) ∗ c

Hence, the operation ∗ is not associative.

(v) For commutativity

a∗ b= $\frac{ab}{4} = \frac{ba}{4} =$ b∗ a

Thus, the operation ∗ is commutative.

For associativity a ∗(b ∗ c) = a∗ $\frac{bc}{4} = \frac{a\frac{bc}{4}}{4} = \frac{abc}{16}$

and (a ∗b) ∗ c = $\frac{ab}{4}$∗c = $\frac{\frac{ab}{4}\cdot c}{4} = \frac{abc}{16}$

∴ a ∗ (b ∗ c) = (a ∗ b) ∗ c Thus, the operation ∗ is associative.

(vi) For commutativity

a ∗ b = ab^2 and b ∗ a = ba^2

∴ a ∗ b ≠ b ∗ a

Thus, the operation ∗ is not commutative.

For associativity

a ∗ (b ∗ c) = a $*(bc^2) = a(bc^2)^2 = ab2c4$

and (a∗ b) ∗ c = $(ab^2)*$ c = $(ab^2)c^2 = ab^2c^2$

∴ a ∗ (b ∗ c) ≠(a ∗ b) ∗ c

Thus, the operation ∗ is not associative.

10. Find which of the operations given above has identity.

SOLUTION

(i) If e is an identity element, then

a $*$ e $=$ a$=$ e $*$ a $\forall a \in Q$

$\Rightarrow$ a$-$e$=$ e$-$a $= \forall a \in Q \Rightarrow$ a$-$ e $=$ a and e $-$ a$=$ a

$\Rightarrow$ e $=$ 0 and e$=$ 2a $\forall a \in Q$

Which is not possible. Hence, identity element does not exist,

(ii) If e is an identity element, then

a $*$ e$=$ a$=$ e$*$ a $\forall a \in Q \Rightarrow a^2 + e^2 = e^2 + a^2 =$ a $\forall a \in Q$.

$\Rightarrow$ e $= \sqrt{a - a^2}\ \forall a \in Q$

Which is not possible. Hence, identity element does not exist,

(iii) If e is an identity element, then

a$*$ e$=$ a$=$ e$*$ $\forall a \in Q$

$\Rightarrow$ a + ae$=$ e + ae$=$ a $\forall a \in Q$.

$\Rightarrow$ a + ae$=$ a and e + ae $=$ a

$\Rightarrow$ e $=$ 0 and e $= \dfrac{a}{1+a}\ \forall a \in Q$

Which is not possible. Hence, identity element does not exist,

(iv) If e is an identity element, then

a$*$e$=$ a$=$e$*$ a $\forall a \in Q$

$\Rightarrow (a - e)^2 =$ (e$-a)^2 =$ a $\forall a \in Q. \Rightarrow$ (a$-e)^2 =$ a and (e$-a)^2 =$ a

$\Rightarrow$ a $-e = \pm\sqrt{a}$ and e $-$ a $= \pm\sqrt{a} \Rightarrow$ e $= a \pm \sqrt{a}$ and e $= a \pm \sqrt{a}\ \forall a \in Q$

Which is not possible. Hence, there is no identity element.

(v) If e is an identity element, then

a $*$ e$=$ a$=$ e $*$ a $\forall a \in Q$.

$\Rightarrow \dfrac{ae}{4} = \dfrac{ea}{4} = a\ \forall a \in Q$.

So, 4 is the identity element.

(vi) If e an identity element, then a$*$ e$=$ a$=$e $*$ a $\forall a \in Q. \Rightarrow e^2 = ea^2 =$ a . $\Rightarrow ae^2 =$ a and $ea^2 =$ a

$\Rightarrow e = \pm 1$ and e $= \dfrac{1}{a}$

Which is not possible. Hence, identity element does not exist.

11. Let A$=$N $\times$ N and $*$ be the binary operation on A defined by (a, b) $*$ (c, d) $=$ (a + c, b + d)

Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

SOLUTION

A $=$ N $\times$ N and $*$ is a binary operation defined on A.

For commutativity

(a, b) $*$ (c, d)$=$ (a + c, b + d) $=$ (c + a, d + b) $=$ (c, d)$*$ (a, b)

The operation $*$ is commutative.

For associativity

[(a, b) $*$ (c, d)]$*$ (e, f)$=$ (a + c, b + d)$*$ (e,f) $=$ (a + c + e, b + d + f)

Also, (a, b) $*$ [(c, d)$*$ (e, f)] $=$ (a, b) $*$ (c + e, d + f)

$=$ (a + c + e, b + d + f)

$\therefore$ [$(a, b)*$ (c, d)] $*$ (e, f) $=$ (a, b) $*$ [(c, d) $*$ (e, f)]

Hence, the operation $*$ is associative.

For identity

Let identity function be (e, f) $\therefore$ (a, b) $*$ (e, f) = (a, b)

(a + e, b + f) = (a, b) $\Rightarrow$ a + e = a, b + f = b

$\Rightarrow$ e = 0, f = 0. But, $0 \notin$ N

Hence, identity element does not exist.

12. State whether the following statements are true or false. Justify.

    (i) For an arbitrary binary operation $*$ on a set N, a $*$ a = a $\forall a \in Q$

    (ii) If $*$ is a commutative binary operation on N, then a$*$ (b$*$ c)= (c $*$ b)$*$ a

    SOLUTION

    (i) False. A binary operation on N is defined as :

    a $*$ a = a $\forall a \in Q$. For example, a $*$ b=a + b $\forall a, b \in Q$, then

    a $*$ a = a + a = 2a$\neq$a.

    Here $'*'$ is not defined.

    (ii) True

    a $*$(b $*$c)=(b $*$c)$*$a =(c $*$ b) $*$a. (b $*$c=c$*$ b is commutative)

13. Consider a binary operation $*$ on N defined as a $*$ b = $a^3 + b^3$. Choose the correct answer.

    (A) Is $*$ both associative and commutative?

    (B) Is $*$ commutative but not associative?

    (C) Is $*$ associative but not commutative?

    (D) Is $*$ neither commutative nor associative?

    SOLUTION

    (B) For commutative

    a $*$ b = $a^3 + b^3 = b^3 + a^3$ = b $*$ a. $\therefore$ $*$ is a commutative operation.

    For associative : a $*$ (b $*$ c) = a $*(b^3 + c^3) = a^3 + (b^3 + c^3)^3$

    and (a $*$ b) $*$ c = $(a^3 + b^3)*$ c = $(a^3 + b^3)^3 + c^3$. $\Rightarrow$ a $*$ (b $*$ c) $\neq$ (a $*$ b) $*$ c. $*$ is not associative.

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