NCERT - Exercise - 1.3

1. Let $f:\{1,\ 3,\ 4\}\to\{1,\ 2,\ 5\}$ and $g:\{1,\ 2,\ 5\}\to\{1,\ 3]$ be given by $f=\{(1,\ 2),\ (3,\ 5),\ (4,\ 1)\}$ and $g=\{(1,\ 3),\ (2,\ 3),\ (5,\ 1)\}$. Write down gof.

   SOLUTION

   $f=\{(1,\ 2),\ (3,\ 5),\ (4,\ 1)\}$ and $g=\{(1,\ 3),\ (2,\ 3),\ (5,\ 1)\}$

   Now, $f(1)=2,\ f(3)=5,\ f(4)=1\ \ and\ \ g(1)=3,\ g(2)=3,\ \ g(5)=1\ (gof)(x)=g[f(x)]\Rightarrow g[f(1)]=g(2)=3$

   $g[f(3)]=g(5)=1,\ \ g[f(4)]=g(1)=3$

   Hence, $gof=\{(1,\ 3),\ (3,\ 1),\ (4,\ 3)\}$.

2. Let f, g and h be functions from R to R. Show that $(f+g)oh=foh+goh$

   SOLUTION

   First, we show that $(f+g)oh=foh+goh$

   L.H.S. $=(f+g)oh=(f+g)[h(x)]=f[h(x)]+g[h(x)]=foh+goh=R.H.S.$ Now, we show that $(f\cdot g)oh=(foh)\cdot(goh)$

   L.H.S. $=(f\cdot g)oh=(f\cdot g)[h(x)]=f[h(x)]\cdot g[h(x)]=(foh)\cdot(goh)=R.H.S.$

3. Find gof and fog, if

   (i) $f(x)=\ |x|\ \ and\ \ g(x)=\ |5x-2|$

   (ii) $f(x)=8x^3\ \ and\ \ g(x)=x^{1/3}$.

   SOLUTION

   (i) $f(x)=\ |x|\ \ and\ \ g(x)=\ |5x-2|\ gof=g(f(x))=g(|x|)=\ |5|x|-2,\ fog=f(g(x))=f(|5x-2|)=||\ 5x-2\ ||=|5x-2|$

   (ii) $f(x)=8x^3\ \ and\ \ g(x)=x^{1/3}$

   $gof=g[f(x)]=g(8x^3)=(8x^3)^{1/3}=2x$ and $fog=f(g(x))=f(x^{1/3})=8(x^{1/3})^3=8x$.

4. If $f(x)=\dfrac{(4x+3)}{(6x-4)},\ x\neq\dfrac{2}{3}$, show that $fof(x)=x$, for all $x\neq\dfrac{2}{3}$. What is the inverse of f?

   SOLUTION

   $f(x)=\dfrac{4x+3}{6x-4},\ x\neq\dfrac{2}{3}$

   L.H.S. $=fof(x)=f(f(x))$

   $$=f\left[\frac{4x+3}{6x-4}\right]=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

   $$=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x=R.H.S.$$

   Now, $y=\dfrac{4x+3}{6x-4}\Rightarrow 6xy-4y=4x+3$

   $\Rightarrow 6xy-4x=4y+3\Rightarrow x(6y-4)=4y+3$

   $\Rightarrow x=\dfrac{4y+3}{6y-4}\Rightarrow y=\dfrac{4x+3}{6x-4}$

   Hence, inverse of f is f.

5. State with reason whether following functions have inverse

   (i) $f:\{1,\ 2,\ 3,\ 4\}\to\{10\}\ \ with\ \ f=\{(1,\ 10),\ (2,\ 10),\ (3,\ 10),\ (4,\ 10)\}$

   (ii) $g:\{5,\ 6,\ 7,\ 8\}\to\{1,\ 2,\ 3,\ 4\}\ \ with\ \ g=\{(5,\ 4),\ (6,\ 3),\ (7,\ 4),\ (8,\ 2)\}$

(iii) $h : \{2,\ 3,\ 4,\ 5\} \to \{7,\ 9,\ 11,\ 13\}\ with\ h = \{(2,\ 7),\ (3,\ 9),\ (4,\ 11),\ (5,\ 13)\}$

SOLUTION

(i) A function is invertible, if it is one-one and onto.

$f = \{(1,\ 10),\ (2,\ 10),\ (3,\ 10),\ (4,\ 10)\}$

It is many-one function (see in figure(I)). Hence, f has noinverse,

(ii) $\mathrm{g} = \{(5,4),(6,3),(7,4),(8,2)\}$

g is many-one fimction (see in figure (II)). Hence, g has noinverse.

(iii) $\mathrm{h} = \{(2,7),(3,9),(4,11),(5,13)\}$

h is one-one and onto function, (see in figure (III)) Hence, h has an inverse.

6. Show that $f : [-1,\ 1] \to R$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1,\ 1] \to Range\ f$.

(Hint : $For\ y \in\ Range\ f,\ y = f(x) = \dfrac{x}{x+2},\ for\ some\ x\ in\ [-1,\ 1]$, i.e., $x = \dfrac{2y}{(1-y)}$).

SOLUTION

$f : [-1,\ 1] \to R$ is given by $f(x) = \dfrac{x}{x+2},\ x \neq 2$ Let $x_1,\ x_2 \in [-1,\ 1] \Rightarrow f(x_1) = \dfrac{x_1}{x_1+2}\ and\ f(x_2) = \dfrac{x_2}{x_2+2}$

$f(x_1) = f(x_2)$

$\Rightarrow \dfrac{x_1}{x_1+2} = \dfrac{x_2}{x_2+2} \Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$

$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Thus, f is one-one.

Now, $f : [-1,\ 1] \to R$, be given for every $y \in R$ (co-domam of f), there exist $x \in [-1,\ 1]$ (domain of f) such that f(x)= y

$\Rightarrow y = f(x) = \dfrac{x}{x+2}\ for\ some\ x\ in\ [-1,\ 1]$

As, $y = \dfrac{x}{x+2} \Rightarrow yx + 2y = x \Rightarrow 2y = x(1-y) \Rightarrow x = \dfrac{2y}{1-y} \Rightarrow f\left[\dfrac{2y}{1-y}\right] = \dfrac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \dfrac{2y}{2y+2-2y} = y$

$\Rightarrow f(x)$ is onto $\Rightarrow$ f is bijective and hence invertible

Now, $x = \dfrac{2y}{1-y} \Rightarrow f^{-1}(y) = \dfrac{2y}{1-y} \Rightarrow f^{-1}(x) = \dfrac{2x}{1-x}$

7. Consider $f : R \to R$ given by $f(x) = 4x+3$. Show that f is invertible. Find the inverse of f.

SOLUTION

First, we show that f is invertible We know that, $f : R \to R,\ f(x) = 4x+3$

Injectivity

Now, let $x_1,\ x_2 \in R. \quad \therefore\ f(x_1) = 4x_1 + 3\ and\ f(x_2) = 4x_2 + 3$ For one-one, $f(x_1) = f(x_2),\ 4x_1 + 3 = 4x_2 + 3 \Rightarrow x_1 = x_2 \therefore f(x)$ is one-one.

Surjectivity $f : R \to R$, given for every $y \in R$ (co-domain of f) there exist $x \in R$ (domain of f) such that f(x) = y $\therefore y = 4x+3 \Rightarrow x = \dfrac{y-3}{4}$ Now, $f\left(\dfrac{y-3}{4}\right) = 4\left(\dfrac{y-3}{4}\right) + 3 = y \quad \therefore\ f(x) = y$ Thus, f(x) is onto function. Thus, f is bijective and hence invertible.

Now, we find the inverse of f. We have f(x) = y

$\Rightarrow x = f^{-1}(y) = \dfrac{y-3}{4}\ or\ f^{-1}(x) = \dfrac{x-3}{4}$

8. Consider f :$R+ \to [4,\infty)$ given by f(x)= x^2+4. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y)=\sqrt{y-4}$, where R+ is the set of all non-negative real numbers.

SOLUTION

$f: R_+ \to [4,\infty)$ and $f(x)=x^2+4$

Injectivity

Consider $x_1, x_2 \in R$

Now, $f(x_1)=x_1^2+4$ and $f(x_2)=x_2^2+4$ if $f(x_1)=f(x_2) \Rightarrow x_1^2+4=x_2^2+4 \Rightarrow x_1=x_2$ (both $x_1, x_2 > 0$)

$\Rightarrow f(x)$is one-one.

Surjectivity

$f: R_+ \to [4,\infty]$ be given, let $y \in [4,\infty)$ (co-domain of f), then there exist an element $x \in R$. (domain of f) such that f(x) = y

Now, $f(x)=y \Rightarrow y=x^2+4 \Rightarrow x=\sqrt{y-4}$

for f(x) $f(\sqrt{y-4})=(\sqrt{y-4})^2+4=y-4+4=y \Rightarrow f^{-1}(y)=\sqrt{y-4}$ or $f^{-1}(x)=\sqrt{x-4}$

9. Consider $f: R_+ \to [-5,\infty)$ given by f(x)= $9x^2+6x-5$.

Show that f is invertible with $f^{-1}(y)=\left(\dfrac{(\sqrt{y+6})-1}{3}\right)$

SOLUTION

f : $R_+ \to [-5,\infty)$ and $f(x)=9x^2+6x-5$

Injectivity

Let $x_1, x_2 \in R \to f(x_1)=9x_1^2+6x_1-5$and $f(x_2)=9x_2^2+6x_2-5$

If$f(x_1)=f(x_2)$

$\Rightarrow 9x_1^2+6x_1-5=9x_2^2+6x_2-5 \Rightarrow 9x_1^2+6x_1=9x_2^2+6x_2$

$\Rightarrow 9(x_1^2-x_2^2)+6(x_1-x_2)=0 \Rightarrow (x_1-x_2)[9(x_1+x_2)+6]=0$

$\Rightarrow x_1-x_2=0 \Rightarrow x_1=x_2 \therefore f(x)$ is one-one.

Surjectivity

$\Rightarrow f: R_+ \to [-5,\infty)$ is given, let$y \in [-5,\infty)$ (co-domain off) then there exist an element x∈R+ (domain of f), such that f(x) = y

$\Rightarrow y=9x^2+6x-5 \Rightarrow 9x^2+6x-(5+y)=0 \Rightarrow x=\dfrac{-(6)\pm\sqrt{(6)^2+4\times 9(5+y)}}{18}$

$\Rightarrow x=\dfrac{-6\pm 6\sqrt{1+(5+y)}}{18}$

$\Rightarrow x=\dfrac{-6+6\sqrt{y+6}}{18} \Rightarrow x=\dfrac{(\sqrt{y+6})-1}{3} \Rightarrow f(x)=f\left(\dfrac{\sqrt{y+6}-1}{3}\right)$

$=9\left[\dfrac{\sqrt{y+6}-1}{3}\right]^2+6\left[\dfrac{\sqrt{y+6}-1}{3}\right]-5$

$=9\left[\dfrac{y+6+1-2\sqrt{y+6}}{6}\right]+2(\sqrt{y+6}-1)-5$

$=y+7-2\sqrt{y+6}+2\sqrt{y+6}-2-5$

$\Rightarrow f(x)=y$. $\Rightarrow f(x)$ is onto.

Thus, f(x) is bijective and hence invertible.

Now, we show that inverse of f is $\dfrac{(\sqrt{y+6})-1}{3}$

Wehave, $f(x)=y \Rightarrow f^{-1}(y)=x$

$\Rightarrow f^{-1}(y)=\dfrac{(\sqrt{y+6})-1}{3}$ or $f^{-1}(x)=\dfrac{\sqrt{x+6}-1}{3}$

10. Let f : $X \to Y$ be an invertible function. Show that f has unique inverse.

(Hint : suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y, fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f).

SOLUTION

Given f : $X \to Y$be invertible.

Thus, f is one-one and onto and therefore f^{-1} exists.

Let $g_1$and g2 be the two inverses of f. Now for all $y \in Y$, $fog_1(y) = I_Y(y) = fog_2(y)$

$\Rightarrow fog_1(y) = fog_2(y) \Rightarrow f[g_1(y)] = f[g_2(y)] \Rightarrow g_1(y) = g_2(y)$

Hence, f has a unique inverse.

11. Consider f : $\{1,2,3\} \to \{a,b,c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Find f^{-1} and show that if $(f^{-1})^{-1} = f$.

SOLUTION

f= $\{(1,a),(2,6),(3,c)\}$. Clearly/is one-one and onto.

Also, $f^{-1} = \{(0,4),(6,2),(c,3)\}$

$\Rightarrow f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$

Also, $(f^{-1})^1 = \{(1,a),(2,b),(3,c)\} = f$

Hence, the result $(f^{-1})^{-1} = f$.

12. Let $f : X \to Y$ be an invertible function. Show that the inverse of $f^{-1}$is f, i.e., $(f^{-1})^{-1} = f$.

SOLUTION

We know that $f : X \to Y$

As f is invertible $\Rightarrow$ f is one-one and onto $\Rightarrow f^{-1}$ exists.

Also, $f^{-1}$is one-one and onto $\Rightarrow f^{-1}$ is invertible.

$\Rightarrow (f^{-1})^{-1}$ exists $\Rightarrow (f^{-1})^{-1} = f$

13. If f : R$\to$ R be given by $f(x) = (3-x^3)^{1/3}$, then fof (x) is

(A) $x^{1/3}$

(B) x^3

(C) x

(D) $(3-x^3)$.

SOLUTION

(C)$f : R \to R$ and $f(x) = (3-x^3)^{1/3}$

$\Rightarrow f[f(x)] = [3-\{f(x)\}^3]^{1/3}$

$\Rightarrow f[f(x)] = [3-\{(3-x^3)^{1/3}\}^3]^{1/3}$

$= [3-(3-x^3)]^{1/3} = (3-3+x^3)^{1/3} = x$

14. Let f : $R - \left\{-\frac{4}{3}\right\} \to R$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map g : Range $f \to R - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$

(B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$

(D) $g(y) = \frac{3y}{4-3y}$

SOLUTION

(B) Given : $f : R - \left\{-\dfrac{4}{3}\right\} \to R$.

We have, $f(x) = \dfrac{4x}{3x+4}$

Now, range of f is $R - \left\{-\dfrac{4}{3}\right\}$. Let $y = f(x)$. $\to y = \dfrac{4x}{3x+4}$

$\Rightarrow$ 3xy + 4y = 4x $\Rightarrow$ x(4−3y)=4y

$\Rightarrow x = \dfrac{4y}{4-3y}$ $\therefore f^{-1}(y) = g(y) = \dfrac{4y}{4-3y}$