NCERT - Exercise - 1.2

1. Show that the function $f : R \to R$, defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R, is the set of all non-zero real numbers. Is the result true, if the domain R, is replaced by N with co-domain being same as R, ?

SOLUTION

$f : R \to R$, defined by $f(x) = \frac{1}{x}$

Injectivity

$f(x_1) = \frac{1}{x_1}$ *and* $f(x_2) = \frac{1}{x_2}$

If $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

Thus, f is one-one.

Surjectivity

Since, $f : R_* \to R_*$

Given any element $y \in R_*$ (co-domain of R_*), then there exist any element $x \in R_*$ (domain of R_*) such that

$f(x) = y$ *and we have*, $f(x) = \frac{1}{x} \Rightarrow \frac{1}{x} = y \Rightarrow x = \frac{1}{y} \Rightarrow f\left(\frac{1}{y}\right) = y$

Thus, f is onto. Hence, f is a one-one and onto function.

The result is not true, since if domain R, is replaced by N and co-domain being same R_* , N does not have inverse.

2. Check the injectivity and surjectivity of the following functions :

(i) $f : N \to N$ *given by* $f(x) = x^2$

(ii) $f : Z \to Z$ *given by* $f(x) = x^2$

(iii) $f : R \to R$ *given by* $f(x) = x^2$

(iv) $f : N \to N$ *given by* $f(x) = x^3$

(v) $f : Z \to Z$ *given by* $f(x) = x^3$

SOLUTION

(i) $f : N \to N$ given by $f(x) = x^2$

Injectivity $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2 \therefore f$ is one-one ie.e, f is injective

Surjectivity There are many such numbersof co-domain which have no image in domain N. e.g., 3∈co-domain N, but there is no pre-image in domain of f. Thus, f is not onto i.e., f is not surjective. Hence, f is injective but not surjective.

(ii) $f : Z \to Z$ given by $f(x) = x^2 \Rightarrow Z = \{(0, \pm1, \pm2, \pm3, ...\}$

Injectivity Let $-1,\ 1 \in Z$, $f(-1) = f(1) \Rightarrow 1 = 1$ But, $-1 \neq 1$. $\therefore$ f is not one-one i.e., f is not injective.

Surjectivity There are many such elements belonging to codomain which have no preimage in its domain Z. $2 \in Z\ (co-domain)$. *But* $2^{1/2} \notin Z\ (co-domain)$ $\therefore$ Element 2 has no pre-image in its co-domain Z $\therefore$ f is not onto i.e., f is not surjective. Hence, f is neither injective nor surjective.

(iii) $f : R \to$ *given by* $f(x) = x^2$

Injectivity Let $-1,\ 1 \in R$, $f(-1) = f(1) \Rightarrow 1 = 0$ But, $-1 \neq 1 \Rightarrow f$ is not injective.

Surjectivity -2 belong to co-domain R of f but $\sqrt{-2}$ doesnot belong to domain R of f. $\therefore$ f is not surjective. Hence, f is neither injective nor surjective.

(iv) $f : N \to N$ given by $f(x) = x^3$

Injectivity Let $x_1, x_2 \in N$, $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ i.e., for every $x \in N$, f has a unique image in its co-domain. $\therefore$ f is one-one and thus f is injective.

Surjectivity There are many such members of co-domain of f which do not have pre-image in its domain e.g., 2, 3 etc. $\therefore$ f is not onto and thus f is not surjective. Hence, f is injective but not surjective.

(v) $f : Z \to Z$, given by $f(x) = x^3$. Here Z is the set of integers.
Injectivity $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2 \therefore$ f is one-one and thus it is injective.

Surjectivity Many members ofco-domain of f have no pre-image in itsdomain. e.g., 2 belonging to its co-domain has no pre-image in its domain of Z. Therefore, f is not surjective. Hence, f is injective but not surjective.

3. Prove that the Greatest Integer Function $f : R \to R$, given $f(x) = |x|$, is neither one-one nor onto, where [x] denotes the greatest integer less than or equal to x.

SOLUTION

$f : R \to R,\ f(x) = [x],$

Injectivity We have, for $1 \leq x < 2$, $f(x) = 1$ Thus, f is not one-one.

Surjectivity $f : R \to R$ does not attain non-integral values. $\therefore$ Non-integer points in R do not have their pre images in the domain. $\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

4. Show that Modulus Function $f : R \to R$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|\ is\ -x$, if x is negative.

SOLUTION

$f : R \to R,\ given\ by\ f(x) = |x|$

Where $|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$

Injectivity

Clearly, f contains (-1, 1), (1, 1), (-2, 2), (2, 2) $\cdots$

We have, $f(-1) = f(1) \Rightarrow |-1| = |1| \Rightarrow 1 = 1$

But, $(-1) \neq (1)$

Thus, negative integers are not images of any elements. $\therefore$ f is not one-one.

Surjectivity We have, for $f : R \to R$, $f(x) = |x|$ assumes only non-negative values. So, negative real numbers in R (co-domain) do not have their pre images in R (domain)

$\therefore$ f is not onto. Hence, f is neither one-one nor onto.

5. Show that the Signum Function $f : R \to R$, given by $f(x) = \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$ is neither one-one nor onto.

SOLUTION

$$f(x) = \begin{cases} 1, & if\ \ x > 0 \\ 0, & if\ \ x = 0 \\ -1, & if\ \ x < 0 \end{cases}$$

Injectivity

We have, $f(1) = f(2) = 1$ but $1 \neq 2$ i.e., $f(x_1) = f(x_2) = 1\ \ for\ x > 0\ \ But,\ x_1 \neq x_2$ Also, $f(-2) = f(-3) = -1.\ \ But,\ -2 \neq -3$ i.e., $f(x_1) = f(x_2) = -1\ \ for\ x < 0.\ But,\ x_1 \neq x_2 \Rightarrow$ f is not one-one.

Surjectivity

Except the numbers $-1,\ 0,\ 1$, no other members of co-domain of f has any pre-image in its domain. $\therefore$ f is not onto. Hence, f is neither one-one nor onto.

6. Let A = 1, 2, 3, B = 4, 5, 6, 7 and let f= (1, 4), (2, 5), (3, 6) be a function from A to B.Show that f is one-one.

SOLUTION

A $= \{1,2,3\}, B = \{4,5,6,7\}$ and f $= \{(1,4),(2,5),(3,6)\}$

We have, f(1) = 4, f(2) = 5 and f(3) = 6. Distinct elements of A have distinct images in B. Hence, f is a one-one function.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : R \to R$, defned by $f(x) = 3-4x$

(ii) $f : R \to R$, defined by $f(x) = 1+x^2$

SOLUTION

(i) $f : R \to R$ defined by $f(x) = 3-4x$.

Injectivity $f(x_1) = f(x_2) \Rightarrow 3-4x = 3-4x_2 \Rightarrow -4x_2 = -4x_2 \Rightarrow x_1 = x_2 \therefore$ f is one-one.

Surjectivity

Now, $f : R \to R$ given for everyy∈R(co-domain of f), there exists an element x∈R (domain of f) such that $f(x) = y \Rightarrow y = 3-4x \Rightarrow x = \frac{3-y}{4}$

$\therefore f\left(\frac{3-y}{4}\right) = 3-4\left(\frac{3-y}{4}\right) = 3-3+y = y$ Hence, f is onto.

Thus, f is one-one and onto or bijective function.

(ii) $f : R \to R$ defined by $f(x) = 1+x^2$ Injectivity

Let $x_1,\ x_2 \in R,\ then\ f(x_1) = 1+x_1^2\ and\ f(x_2) = 1+x_2^2\ f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$ Thus, f is not one-one.

Surjectivity

Now, $f : R \to R$, given for every y∈R(co-domain of f), there exists an element x∈R(domain of f) such that f(x) = y ⇒

Thus, elements less than 1 has no pre-image. ∴ f is not onto. Hence, f is neither one-one nor onto and hence not bijective.

8. Let, A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a,\ b) = (b,\ a)$ is bijective function.

SOLUTION

Injectivity

Let $(a_1,\ b_1)\ and\ (a_2,\ b_2) \in A \times B$ such that, $f(a_1,\ b_1) = f(a_2,\ b_2) \Rightarrow (b_1,\ a_1) = (b_2,\ a_2) \Rightarrow b_1 = b_2\ and\ a_1 = a_2 \Rightarrow (a_1,\ b_1) = (a_2,\ b_2)$

Thus, $f(a_1,\ b_1) = f(a_2,\ b_2) \Rightarrow (a_1,\ b_1) = (a_2,\ b_2)$ [For all $(a_1,\ b_1),\ (a_2,\ b_2) \in A \times B$] So, f is an injective function.

Surjectivity

Let (b, a) be an arbitrary element of $B \times A$, where, $b \in B$ and a∈A ⇒ (a, b)∈ $A \times B$ Thus, for all (b, a)∈B x A, their exists (a, b) ∈ (A x B) such that, $f(a,\ b) = (b,\ a)$ So, $f : A \times B \to B \times A$ is an onto function.

Hence, f is a bijective function.

9. Let $f : N \to N$ be defined by $f(n) = \begin{cases} \frac{n+1}{2},\ if\ n\ is\ odd \\ \frac{n}{2},\quad if\ n\ is\ even \end{cases}$ for all $n \in N$. State whether the function f is bijective f is bijective. Justify your answer.

SOLUTION

Injectivity

Here, $f(1) = \frac{1+1}{2} = 1,\ \ f(2) = \frac{2}{2} = 1,\ \ f(3) = \frac{3+1}{2} = 2,\ \ f(4) = \frac{4}{2} = 2$ Thus $f(2k-1) = \frac{(2k-1)+1}{2} = k\ \ and\ \ f(2k) = \frac{2k}{2} = k$

$\Rightarrow f(2k-1) = f(2k),\ \ where\ \ k \in N$

But, $2k-1 \neq 2k$, where $k \in N \Rightarrow f$ is not one-one.

Surjectivity

But, f is onto because range of f = N

$\Rightarrow$ f is onto.

Hence, f is not bijective.

10. Let $A = R - \{3\}$ *and* $B = R - \{1\}$. Consider the function $f: A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

SOLUTION

$A = R - \{3\},\ B = R - \{1\}\ \textit{and}\ f(x) = \dfrac{x-2}{x-3}$

Let $x_1,\, x_2 \in A \Rightarrow\ f(x_1) = \dfrac{x_1-2}{x_1-3}\ \textit{and}\ f(x_2) = \dfrac{x_2-2}{x_2-3}$

Injectivity

$f(x_1) = f(x_2) \Rightarrow \dfrac{x_1-2}{x_1-3} = \dfrac{x_2-2}{x_2-3} \Rightarrow (x_1-2)(x_2-3) = (x_2-2)(x_1-3) \Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is a one-one function.

Surjectivity

$f: A \to B$, let $y \in B$ (co-domain of f) be any element, then there exist $x \in A$ (domain of f) such that $f(x) = y \Rightarrow y = \dfrac{x-2}{x-3} \Rightarrow$ $y(x-3) = x-2 \Rightarrow xy - 3y = x - 2$

$\Rightarrow x(y-1) = 3y - 2 \Rightarrow x = \dfrac{3y-2}{y-1}$

$$\therefore f\left(\frac{3y-2}{y-1}\right) = \frac{\dfrac{3y-2}{y-2} - 2}{\dfrac{3y-2}{y-1} - 3} = \frac{3y-2-2y+2}{3y-2-3y+3} = y$$

$\therefore$ f is an onto function.

Hence, f is one-one and onto function.

11. Let $f: R \to R$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

SOLUTION

(D) $f: R \to R$ defined as $f(x) = x^4$.

Injectivity

Let $x_1,\, x_2 \in R,\quad f(x_1) = f(x_2) \Rightarrow x_1^4 = x_2^4 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$ $\therefore$ f is not one-one.

Surjectivity

$f: R \to R$. Let, $y \in R$ (co-domain of f) be any element, then there exist $x \in R$ (domain of f) such that $f(x) = y \Rightarrow y = x^4 \Rightarrow x = \pm\, y^{1/4}$

Now, $f(y^{1/4}) = (y^{1/4})^4 = y$ and $f(-y^{1/4}) = (-y^{1/4})^4 = -y$ $\therefore$ f is not onto.

Thus, f is neither one-one nor onto.

12. Let $f: R \to R$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

SOLUTION

(A) Injectivity

Let $x_1,\ x_2 \in R$ such that $f(x_1) = f(x_2), \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

Surjectivity

For any $y \in R$ (co-domain of f), there exist $x \in R$ (domain of f) such that $f(x) = y \Rightarrow 3x = y \Rightarrow x = \dfrac{y}{3}$

$\Rightarrow f(x) = f\left(\dfrac{y}{3}\right) = 3.\dfrac{y}{3} = y \Rightarrow$ f is onto Thus, f is one-one and onto.