NCERT - Miscellaneous Exercise

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

SOLUTION

Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ Expanding by R_1, we get $= x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta \begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$

$= x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta) = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x(1) = -x^3$ which is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

SOLUTION

Let $\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

MultiplyingR_1, R_2 and R_3 by a, b and c respectively and dividing the determinant by abc, we get $\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

Taking abc common from C_3, we get $\Delta = \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ c^2 & 1 & c^3 \end{vmatrix}$

Interchanging $C_1 \leftrightarrow C_2$,we get $\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Hence, $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

SOLUTION

$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along R_1, we got $= \cos\alpha\cos\beta \begin{vmatrix} \cos\beta & 0 \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} - \cos\alpha\cos\beta \begin{vmatrix} -\sin\beta & 0 \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} - \sin\alpha \begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\sin\beta & \sin\alpha\sin\beta \end{vmatrix}$

$= \cos\alpha\cos\beta(\cos\beta\cos\alpha) - \cos\alpha\sin\beta(-\sin\beta\cos\alpha) - \sin\alpha(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta)$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta = \cos^2\alpha \cdot 1 + \sin^2\alpha \cdot 1 = \cos^2\alpha + \sin^2\alpha = 1.$

4. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, show that either $a+b+c = 0$or $a = b = c$.

SOLUTION

$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$ Taking $2(a+b+c)$ common from C_1, we get

$\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$ Applying $R_2 \to R_2 - R_3$ and $R_3 \to R_3 - R_1$, we get $\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & a-c & b-a \\ 0 & b-a & c-b \end{vmatrix}$

Expanding along C_1,we get $\Delta = 2(a+b+c)[(a-c)(c-b)-(b-a)(b-a)]$

$= 2(a+b+c)[ac-ab-c^2+cb-(b^2+a^2-2ba)]\,2(a+b+c)(ac-ab-c^2+cb-b^2-a^2+2ab) = -2(a+b+c)[a^2+b^2+c^2-ab-$

$= -(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

When, $\Delta = 0 \Rightarrow -(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] = 0 \Rightarrow a+b+c = 0$ or $(a-b)^2, (b-c)^2, (c-a)^2 = 0 \Rightarrow a+b+c = 0$ or $a = b, b = c, c = a \Rightarrow a+b+c = 0$ or $a = b = c$

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

SOLUTION

$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $\Rightarrow \begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$

$\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get $\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_3$, we get $\Rightarrow (3x+a)\begin{vmatrix} 1 & 0 & x \\ 0 & a & 0 \\ 0 & -a & a \end{vmatrix} = 0$

Expanding along R_1, we get $(3x+a)(a^2) = 0$ Since $a \neq 0 \Rightarrow 3x+a = 0 \Rightarrow x = \dfrac{-a}{3}$.

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+bc & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

SOLUTION

L.H.S. : $= \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+bc & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking $a, b,$ ccommon from $C_1, C_2 \& C_3$, we get $= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$

Applying $C_3 \to C_3 - (C_1 + C_2)$, we get $= abc\begin{vmatrix} a & c & 0 \\ a+b & b & -2b \\ b & b+c & -2b \end{vmatrix}$

Taking$-2b$ common from C_3, we get $= -2b(abc)\begin{vmatrix} a & c & 0 \\ a+b & b & 1 \\ b & b+c & 1 \end{vmatrix}$

Expanding the determinant along R_1, we get $= -2b(abc)[a(b-b-c)-c(a+b-c)]. = -2b(abc)[-2ac] = 4a^2b^2c^2 = R.H.S$
Hence, proved.

7. If $A^{-1}=\begin{bmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{bmatrix}$ and $B=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$ find $(AB)^{-1}$.

SOLUTION

Given, $A^{-1}=\begin{bmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{bmatrix}$, $B=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$

Now, $|B|=1(3)-2(-1)-2(2)=3+2-4=1\neq 0 \therefore B^{-1}$ exists. Since $(AB)^{-1}=B^{-1}A^{-1}$, so, we need to calculate B^{-1}. Cofactors of elements of B are :

$B_{11}=(-1)^{1+1}\begin{vmatrix}3&0\\-2&1\end{vmatrix}=3,$

$B_{12}=(-1)^{1+2}\begin{vmatrix}-1&0\\0&1\end{vmatrix}=1,$

$B_{13}=(-1)^{1+3}\begin{vmatrix}-1&3\\0&-2\end{vmatrix}=2,$

$B_{21}=(-1)^{2+1}\begin{vmatrix}2&-2\\-2&1\end{vmatrix}=2,$

$B_{22}=(-1)^{2+2}\begin{vmatrix}1&-2\\0&1\end{vmatrix}=1,\ B_{23}=(-1)^{2+3}\begin{vmatrix}1&2\\0&-2\end{vmatrix}=2,,$

$B_{31}=(-1)^{3+1}\begin{vmatrix}2&-2\\3&0\end{vmatrix}=6,\ B_{32}=(-1)^{3+2}\begin{vmatrix}1&-2\\-1&0\end{vmatrix}=2,\ B_{33}=(-1)^{3+3}\begin{vmatrix}1&2\\-1&3\end{vmatrix}=5$

$\therefore adj\,B$

$=\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}$

Hence, $B^{-1}=\frac{1}{|B|}(adj\,B)=\frac{1}{1}\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}$

Now, $B^{-1}A^{-1}=\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}\begin{bmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{bmatrix}$

$=\begin{bmatrix}9-30+30&-3+12-12&3-10+12\\3-15+10&-1+6-4&1-5+4\\6-30+25&-2+12-10&2-10+10\end{bmatrix}=\begin{bmatrix}9&-3&5\\-2&1&0\\1&0&2\end{bmatrix}$

8. Let $A=\begin{bmatrix}1&-2&1\\-2&3&1\\1&1&5\end{bmatrix}$. Verify that,

(i) $[adj\,A]^{-1}=adj(A^{-1})$

(ii) $(A^{-1})^{-1}=A$

SOLUTION

$A=\begin{bmatrix}1&-2&1\\-2&3&1\\1&1&5\end{bmatrix}$

Now, $|A|=1(15-1)+2(-10-1)+1(-2-3)$

$=14-22-5=-13\neq 0 \therefore A^{-1}$ exists.

Cofactors of elements of A are : $A_{11}=(-1)^{1+1}\begin{vmatrix}3&1\\1&5\end{vmatrix}=15-1=14,\ A_{12}=(-1)^{1+2}\begin{vmatrix}-2&1\\1&5\end{vmatrix}=-(-10-1)=11,$

$A_{13}=(-1)^{1+3}\begin{vmatrix}-2&3\\1&1\end{vmatrix}=(-3-2)=-5,\ A_{21}=(-1)^{2+1}\begin{vmatrix}-2&1\\1&5\end{vmatrix}=-(-10-1)=11,$

$A_{22}=(-1)^{2+2}\begin{vmatrix}1&1\\1&5\end{vmatrix}=5-1=4,\ A_{23}=(-1)^{2+3}\begin{vmatrix}1&-2\\1&1\end{vmatrix}=-(1+2)=-3,,$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -2-3 = -5,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1+2) = -3, A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = 3-4 = -1$

So, $A^{-1} = \frac{1}{|A|}(adj\,A) = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$

(i) Now, $|adj\,A| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$

$= 14(-4-9) - 11(-11-15) - 5(-33+20) = -182+286+65 = 269 \neq 0 \therefore$ adj A is invertible and $(adj\,A)^{-1} = \frac{1}{|adj\,A|}\{adj\,(adj\,A)\}$

Now, to obtain $adj(adj\,A)$, the cofactors of adj A are : $adjA_{11} = (-1)^{1+1}\begin{vmatrix} 4 & -3 \\ -3 & -1 \end{vmatrix} = -4-9 = -13$

$adjA_{12} = (-1)^{1+2}\begin{vmatrix} 11 & -3 \\ -5 & -1 \end{vmatrix} = -(-11-15) = 26, adjA_{13} = (-1)^{1+3}\begin{vmatrix} 11 & 4 \\ -5 & -3 \end{vmatrix} = -33+20 = -13,$

$adjA_{21} = (-1)^{2+1}\begin{vmatrix} 11 & -5 \\ -3 & -1 \end{vmatrix} = -(-11-15) = 26, adjA_{22} = (-1)^{2+2}\begin{vmatrix} 14 & -5 \\ -5 & -1 \end{vmatrix} = -14-25 = -39,$

$adjA_{23} = (-1)^{2+3}\begin{vmatrix} 14 & 11 \\ -5 & -3 \end{vmatrix} = -(-42+55) = -13,,$

$adjA_{32} = (-1)^{3+2}\begin{vmatrix} 14 & -5 \\ 11 & -3 \end{vmatrix} = -(-42+55) = -13, adjA_{33} = (-1)^{3+3}\begin{vmatrix} 14 & 11 \\ 11 & 4 \end{vmatrix} = (56-121) = -65$

So, $(adj\,A)^{-1} = \frac{1}{|adj\,A|}\{adj(adjA)\} = \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ ..(1)

Also, $adj(A^{-1}) = adj\left(\frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}\right)$

Similarly, $adj\begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix} = \begin{bmatrix} -\frac{13}{169} & \frac{26}{169} & -\frac{13}{169} \\ \frac{26}{169} & -\frac{39}{169} & -\frac{13}{169} \\ -\frac{13}{169} & -\frac{13}{169} & -\frac{65}{169} \end{bmatrix}$

$\Rightarrow (adj\,A^{-1}) = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ ....(2)

From (1) and (2), we find that, $adj(A^{-1}) = (adjA)^{-1}$.

(ii) $|A^{-1}| = \left|\frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}\right| = \frac{1}{(13)^2}\{-14(-4-9)+11(-11-15)+5(-33+20)\}$

$= \frac{1}{13\times 13\times 13}(-169) = -\frac{1}{13} \neq 0. \therefore (A^{-1})^{-1}$ exists and $(A^{-1})^{-1} = \frac{1}{|A^{-1}|}(adj\,A^{-1})$

$= \frac{1}{-\frac{1}{13}}\cdot\frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ (using (2)) = A.

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

SOLUTION

$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $\begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix}$

Taking $2(x+y)$common from C_1, we get $2(x+y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get $2(x+y)\begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$

Expanding along C_1, we get $2(x+y)[x(-x)-(-y)(x-y)] = 2(x+y)[-x^2+xy-y^2] = -2(x+y)(x^2-xy+y^2) = -2(x^3+y^3)$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

SOLUTION

Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$

Expanding along C_1, we get $1 \times y \times x = xy$. Using properties of determinants in questions 11 to15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

SOLUTION

L.H.S.$= \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Applying $C_3 \to C_3 + C_1$, we get L.H.S.$= = \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$

Taking $(\alpha+\beta+\gamma)$ common from C_3, we get L.H.S. $= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get L.H.S. $=(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\beta & \gamma^2-\beta^2 & 0 \end{vmatrix}$

Expanding along C_3, we get L.H.S. $= (\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & \beta^2-\alpha^2 \\ \gamma-\beta & \gamma^2-\beta^2 \end{vmatrix}$

Taking $(\beta-\alpha)$ and $(\gamma-\beta)$ common from R_1 and R_2 respectively, we get $L.H.S. = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\beta)\begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\beta \end{vmatrix}$

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = R.H.S.$ Hence, proved.

12. . $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$

SOLUTION

Let $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$ Using property 5, we get

$$\Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Takingx, y, z and p common from R_1, R_2, R_3 and C_3in determinant II.

Interchanging $C_2 \leftrightarrow C_3$,in determinant I. $\Delta = -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Interchanging $C_1 \leftrightarrow C_2$, in determinant I. $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get $\Delta = (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$ Taking $(y-x)$ and $(z-x)$ common from R_2 and R_3, we get $\Delta = (1+pxyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 1 & z+x \end{vmatrix}$

Applying $R_2 \to R_2 - R_3$, we get $\Delta = (1+pxyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 0 & y-z \\ 0 & 1 & z+x \end{vmatrix}$

Expanding along C_1, we get $\Delta = (1+pxyz)(y-x)(z-x)\{0-(y-z)\} = (1+pxyz)(x-y)(y-z)(z-x)$. Hence, proved.

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

SOLUTION

Let $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

Taking $(a+b+c)$ common from C_1, we get $\Delta = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get $\Delta = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$

Expanding along C_1, we get $\Delta = (a+b+c)[(2b+a(2c+a)-(a-c)(a-b)] = (a+b+c)(4bc+2ab+2ac+a^2-a^2+ab+ac-bc)$ $= (a+b+c)(3ab+3bc+3ca) = 3(a+b+c)(ab+bc+ca)$ Hence, proved.

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

SOLUTION

L.H.S. $= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get L.H.S. $= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$

Expanding along C_1, we get $= 1(7+3p-6-3p) = 1(1) = 1 =$ R.H.S. Hence, proved.

15. . $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

SOLUTION

Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

Then, using $\cos(A+B) = cosAcosB - sinAsinB$, we get $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$

Applying $C_3 \to C_3 + (\sin\delta)C_1 - (\cos\delta)C_2$, we get $= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0$ $(As,\ C_3 = 0)$ Hence, proved.

16. Solve the system of the following equations $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

SOLUTION

The equations can be written in the form AX=B, where, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36) = 2(75) - 3(-110) + 10(72)\ 150 + 330 + 720 = 1200 \neq 0 \therefore A^{-1}$ exists.

Now, let A_{ij} be the cofactor of the element in i^{th} row and j^{th} column. The cofactors are : $A_{11} = (-1)^{1+1} \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 120 - 45 = 75$ $A_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -(-80-30) = 110.$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 36 + 36 = 72,\ A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -(-60-90) = 150$ $A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = -40 - 60 = -100,$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -(18-18) = 0,\ A_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 15 + 60 = 75,\ A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -(10 - 40) = 30,\ A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -12 - 12 = -24$

$\therefore adj\,A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix} = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}(adj\,A) = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

As, $AX = B \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

Thus, $\frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$ Hence, $x = 2, y = 3, z = 5$

Choose the correct answer in questions 17 to 19.

17. If a, b, c, are in A.P, then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is

(A) 0

(B) 1

(C) x

(D) 2x

SOLUTION

(A) : $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$, we get $\begin{vmatrix} -1 & -1 & 2a-2b \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, we get $\begin{vmatrix} 0 & -1 & 2a-2b \\ -1 & x+4 & x+2b \\ -1 & x+5 & x+2c \end{vmatrix}$

Expanding along R_1, we get $1\begin{vmatrix} -1 & x+2b \\ -1 & x+2c \end{vmatrix} + (2a-2b)\begin{vmatrix} -1 & x+4 \\ -1 & x+5 \end{vmatrix} = -x-2c+x+2b+(2a-2b)[-x-5+x+4]$

$= 2b-2c+(2a-2b)(-1) = 2b-2c-2a+2b = 2[2b-(c+a)] = 2\left[2\left(\frac{a+c}{2}\right)-(c+a)\right] = 0$ [Since a, b, c are in A.P.]

18. If x, y, z are non-zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

(D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SOLUTION

(A) : Let A= $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $\therefore |A| = xyz \neq 0, A^{-1}$ exists. Now, cofactors of elements of A are :

$A_{11} = (-1)^{1+1}\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = yz, A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = 0\ A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0, A_{22} = (-1)^{2+2}\begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = xz,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0, A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ y & 0 \end{vmatrix} = 0,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0,$

$A_{33} = (-1)^{3+3}\begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy$

$\therefore adj\,A$

$= \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

19. Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then,

(A) $Det(A) = 0$

(B) $Det(A) \in (2, \infty)$

(C) $Det(A) \in (2, 4)$

(D) $Det(A) \in [2, 4]$

SOLUTION

(D) : $\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ Expanding along R_1, we get $1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1) = 1 + \sin^2\theta + 1 + \sin^2\theta = 2(1+\sin^2\theta)$

As, $\sin^2\theta \in [0,1] \Rightarrow 1 + \sin^2\theta \in [1,2]$

$\Rightarrow 2(1+\sin^2\theta) \in [2,4]$

Our Mathematics E-Books

1. J.E.E. (Join Entrance Exam)
   ★ Chapter Tests (Full Syllabus- Fully Solved)
   ★ Twenty Mock Tests (Full Length - Fully Solved)
2. B.I.T.S.A.T. Twenty Mock Tests (Fully Solved)
3. C.BS.E.
   ★ Work-Book Class XII (Fully Solved)
   ★ Objective Type Questions Bank C.B.S.E. Class XII (Fully Solved)
   ★Chapter Test Papers Class XII (Fully Solved)
   ★ Past Fifteen Years Topicwise Questions (Fully Solved)
   ★ Sample Papers Class XII (Twenty Papers Fully Solved- includes 2020 solved paper)
   ★ Sample Papers Class X (Twenty Papers Fully Solved -includes 2020 solved paper)
4. I.C.S.E. & I.S.C.
   ★Work-Book Class XII (Fully Solved)
   ★ Chapter Test Papers Class XII (Fully Solved)
   ★ Sample Papers Class XII (Twenty Papers Fully Solved -includes 2020 solved paper)
   ★ Sample Papers Class X (Twenty Papers Fully Solved -includes 2020 solved paper)
5. Practice Papers for SAT -I Mathematics (15 Papers - Fully Solved)
6. SAT - II Subject Mathematics (15 Papers - Fully Solved)