0.1 Definition of Differentiation

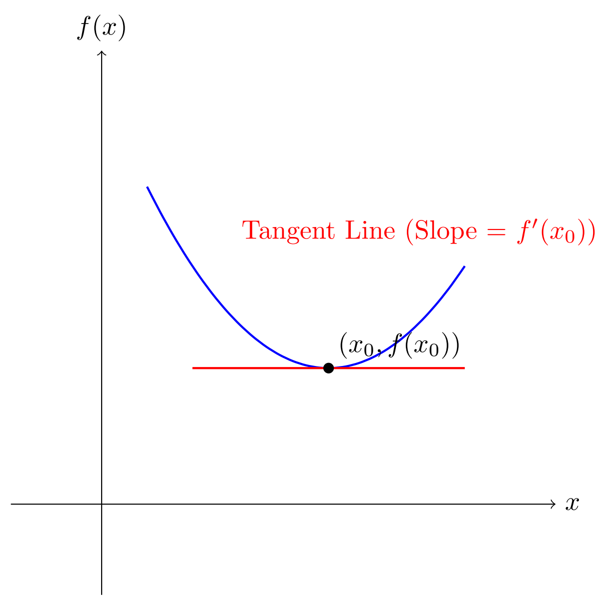
Differentiation is the process of finding the derivative of a function, which represents the rate of change of the function with respect to a variable. Mathematically, the derivative of a function $f(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This definition expresses the instantaneous rate of change of $f(x)$ at a given point x.

Instantaneous Rate of Change

This definition expresses the instantaneous rate of change of $f(x)$ at a given point x. The derivative $f'(x)$ represents the slope of the tangent line to the curve at x.



Formulas and Properties of Differentiation

1. Definition of Differentiation

The derivative of a function $f(x)$ is given by the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This represents the instantaneous rate of change of $f(x)$ at a given point x.

2. Basic Derivatives

$$\frac{d}{dx}(c) = 0 \quad \text{(where } c \text{ is a constant)}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$$

3. Trigonometric Derivatives

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad x \neq \frac{\pi}{2} + n\pi$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x, \quad x \neq n\pi$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \quad x \neq \frac{\pi}{2} + n\pi$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x, \quad x \neq n\pi$$

4. Logarithmic and Exponential Derivatives

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \ln a, \quad a > 0$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad x > 0, a > 0, a \neq 1$$

5. Differentiation Rules

(i) Sum and Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

(ii) Product Rule

$$\frac{d}{dx}[u(x)v(x)] = u'v + uv'$$

(iii) Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}, \quad v(x) \neq 0$$

(iv) Chain Rule

If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

6. Higher Order Derivatives

The second derivative (acceleration in physics) is:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The nth derivative is:

$$f^{(n)}(x) = \frac{d^n y}{dx^n}$$

7. Implicit Differentiation

For an equation of the form $F(x, y) = 0$, differentiate both sides with respect to x, treating y as an implicit function of x.

Example: Given $x^2 + y^2 = 25$,

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

8. Parametric Differentiation

For a curve given by parametric equations $x = f(t), y = g(t)$, the derivative is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

9. Logarithmic Differentiation

If $y = f(x)^{g(x)}$, take the logarithm on both sides:

$$\ln y = g(x) \ln f(x)$$

Differentiate using the chain rule.

10. Differentiation of Inverse Functions

If $y = f^{-1}(x)$, then:

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

11. Applications of Differentiation

- **Finding Tangents and Normals:** The slope of the tangent line at x is $f'(x)$, and the normal's slope is $-\frac{1}{f'(x)}$.
- **Maxima and Minima:** If $f'(x) = 0$ and $f''(x) > 0$, it's a local minimum; if $f''(x) < 0$, it's a local maximum.
- **Rate of Change:** $f'(x)$ gives the instantaneous rate of change of a quantity.

0.2 Examples of Differentiation

0.2.1 Example 1: Derivative of a Polynomial Function

Let $f(x) = x^2 + 3x + 5$. Using the first principles,

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) + 5 - (x^2 + 3x + 5)}{h}$$

Expanding,

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h + 5 - x^2 - 3x - 5}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h}$$

$$= \lim_{h\to 0} (2x + h + 3)$$

$$= 2x + 3$$

Thus,

$$\frac{d}{dx}(x^2 + 3x + 5) = 2x + 3$$

0.2.2 Example 2: Derivative of Trigonometric Functions

Let $f(x) = \sin x$, then using first principles,

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Using the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$,

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

Since $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$,

$$= \cos x$$

Thus,

$$\frac{d}{dx}(\sin x) = \cos x$$

0.3 Methods of Differentiation

There are several techniques for differentiation:

0.3.1 1. Power Rule

If $f(x) = x^n$, then:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Example:

$$\frac{d}{dx}x^5 = 5x^4$$

0.3.2 2. Product Rule

If $f(x) = u(x)v(x)$, then:

$$\frac{d}{dx}[uv] = u'v + uv'$$

Example:

$$\frac{d}{dx}[x^2 \sin x] = 2x \sin x + x^2 \cos x$$

0.3.3 3. Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$, then:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Example:

$$\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{1 \cdot \sin x - x \cos x}{\sin^2 x}$$

0.3.4 4. Chain Rule

If $f(x) = g(h(x))$, then:

$$\frac{d}{dx}f(x) = g'(h(x)) \cdot h'(x)$$

Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$