PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. (a) $\sin^{-1}(\sin x) = x,\ \forall x \in \left[\dfrac{-\pi}{2},\ \dfrac{\pi}{2}\right]$

   (b) $\cos^{-1}(\cos x) = x,\ \forall x \in [0,\ \pi]$

   (c) $\tan^{-1}(\tan x) = x,\ \forall x \in \left(\dfrac{-\pi}{2},\ \dfrac{\pi}{2}\right)$

   (d) $\text{cosec}^{-1}(\text{cosec}x) = x,\ \forall x \in \left[\dfrac{-\pi}{2},\ \dfrac{\pi}{2}\right],\ x \neq 0$

   (e) $\sec^{-1}(\sec x) = x,\ \forall x \in [0,\ \pi],\ x \neq \dfrac{\pi}{2}$

   (f) $\cot^{-1}(\cot x) = x,\ \forall x \in (0,\ \pi)$

---

2. (a) $\sin(\sin^{-1}x) = x,\ \forall x \in [-1,\ 1]$

   (b) $\cos(\cos^{-1}x) = x,\ \forall x \in [-1,\ 1]$

   (c) $\tan(\tan^{-1}x) = x,\ \forall x \in (-\infty,\ \infty)$

   (d) $\text{cosec}(\text{cosec}^{-1}x) = x,\ \forall x \in (-\infty,\ -1] \cup [1,\ \infty)$

   (e) $\sec(\sec^{-1}x) = x,\ \forall x \in (-\infty,\ -1] \cup [1,\ \infty)$

   (f) $\cot(\cot^{-1}x) = x,\ \forall x \in (-\infty,\ \infty)$

---

3. (a) $\sin^{-1}(-x) = -\sin^{-1}x,\ \forall x \in [-1,\ 1]$

   (b) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ \forall x \in [-1,\ 1]$

   (c) $\tan^{-1}(-x) = -\tan^{-1}x,\ \forall x \in (-\infty,\ \infty)$

   (d) $\text{cosec}^{-1}(-x) = \text{cosec}^{-1}x,\ \forall x \in (-\infty,\ -1] \cup [1,\ \infty)$

   (e) $\sec^{-1}(-x) = \pi - \sec^{-1}x,\ \forall x \in (-\infty,\ -1] \cup [1,\ \infty)$

   (f) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ \forall x \in (-\infty, \infty)$

---

4. (a) $\sin^{-1}\left(\dfrac{1}{x}\right) = \text{cosec}^{-1}x,\ \forall x \in (-\infty,\ -1] \cup [1, \infty)$

   (b) $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}x,\ \forall x \in (-\infty, -1] \cup [1,\ \infty)$

   (c) $\tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1}x, & for\ x > 0 \\ -\pi + \cot^{-1}x, & for\ x > 0 \end{cases}$

---

5. (a) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2},\ \forall x \in [-1,\ 1]$

   (b) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},\ \forall x \in (-\infty, \infty)$

   (c) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2},\ \forall x \in (-\infty,\ 1] \cup [1,\ \infty)$

---

6. (a) $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & if\ \ xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & if\ \ x > 0,\ y > 0\ \ and\ \ xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & if\ x < 0,\ y < 0\ \ and\ \ xy > 1 \end{cases}$

(b) $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ x>0,\ y<0\ and\ xy<-1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ x<0,\ y>0\ and\ xy<-1 \end{cases}$

7. (a) $\sin^{-1}x + \sin^{-}y = \begin{cases} \sin^{-}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}. \\ if\ -1 \le x,\ y \le 1\ and\ x^2+y^2 \le 1 \\ or \\ if\ xy<0\ and\ x^2+y^2>1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \\ if\ 0<x,\ y\le 1\ and\ x^2+y^2>1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \\ if\ -1 \le x,\ y<0\ and\ x^2+y^2>1 \end{cases}$

(b) $\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \\ if\ -1 \le x,\ y \le 1\ and\ x^2+y^2 \le 1 \\ or \\ if\ xy>0\ and\ x^2+y^2>1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \\ if\ 0<x\le 1,\ -1\le y\le 0\ and\ x^2+y^2>1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \\ if\ -1 \le x<0,\ 0<y\le 1\ and\ x^2+y^2>1 \end{cases}$

8. (a) $\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \\ if\ -1\ \le x,\ y \le 1\ and\ x+y \ge 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \\ if\ -1 \le x,\ y \le 1\ and\ x+y \le 0 \end{cases}$

(b) $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \\ if\ -1 \le x,\ y \le 1\ and\ x \le 1\ and\ x \le y \\ -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \\ if\ -1 \le y \le 0,\ 0<x\le 1\ and\ x \ge y \end{cases}$

9. (a) $2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & if\ -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & if\ -\dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & if\ -1 \le x \le -\dfrac{1}{\sqrt{2}} \end{cases}$

(b) $3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), & if\ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), & if\ \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & if\ -1 \le x < -\dfrac{1}{2} \end{cases}$

10. (a) (i) $2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1), & if\ 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), & if\ -1 \le x \le 0 \end{cases}$

(b) $3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), & if\ \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), & if\ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), & if\ -1 \le x \le -\dfrac{1}{2} \end{cases}$

11. (a) $2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & if -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & if\ x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & if\ x < -1 \end{cases}$

(b) $3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & if\ -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & if\ x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & if\ x < -\dfrac{1}{\sqrt{3}} \end{cases}$

12. (a) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & if\ -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & if\ x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & if\ x < -1 \end{cases}$

(b) $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & if\ 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & if\ -\infty < x \le 0 \end{cases}$

13. (a) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\dfrac{1}{x}$

(b) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\dfrac{1}{x} = \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

(c) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\dfrac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

14. $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1}x) = x$, provided that $-1 \le x \le 1$ *and* $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

15. $\cos^{-1}(\cos\theta) = \theta$ and $(\cos^{-1}x) = x$, provided that $-1 \le x \le 1$ *and* $0 \le \theta \le \pi$.

16. $\tan^{-1}(\tan\theta) = \theta$ *and* $\tan(\tan^{-1}x) = x$, provided that $-\infty < x < \infty$ *and* $\dfrac{-\pi}{2} < \theta < \dfrac{\pi}{2}$.

17. $\cot^{-1}(\cot\theta) = \theta$ *and* $\cot(\cot^{-1}x) = x$, provided that $-\infty < x < \infty$ *and* $0 < \theta < \pi$.

18. $\sec^{-1}(\sec\theta) = \theta$ *and* $\sec(\sec^{-1}x) = x$

19. $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$ *and* $\text{cosec}(\text{cosec}^{-1}x) = x$.

20. $\sin^{-1}x = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$ *or* $\text{cosec}^{-1}x = \sin^{-1}\left(\dfrac{1}{x}\right)$

21. $\cos^{-1}x = \sec^{-1}\left(\dfrac{1}{x}\right)$ *or* $\sec^{-1}x = \cos^{-1}\left(\dfrac{1}{x}\right)$

22. $\tan^{-1}x = \cot^{-1}\left(\dfrac{1}{x}\right)$ *or* $\cot^{-1}x = \tan^{-1}\left(\dfrac{1}{x}\right)$

23. $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$

24. $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\dfrac{1}{x} = \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

25. $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

26. $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2},\ where\ -1 \le x \le 1\ \tan^{-1}x+\cot^{-1}x = \frac{\pi}{2},\ where\ -\infty < x < \infty\ \sec^{-1}x+\text{cosec}^{-1}x = \frac{\pi}{2},\ where\, x \le -1\ or\, x \ge 1$

27. $\tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),\ if\ xy < 1\ \tan^{-1}x+\tan^{-1}y = \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right),\ if\ xy > 1\ \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1-xy}\right)$

28. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})\ if\ x,\ y \ge 0$ and $x^2+y^2 \le 1$

29. $\sin^{-1}x \pm \sin^{-1y} = \pi - \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$ if $x,\ y \ge 0,\ x^2+y^2 > 1$

30. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$, if $x,\ y \ge 0\ \ and\ x^2+y^2 \le 1$

31. $\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}(xy \pm \sqrt{1-x^2}\sqrt{1-y^2}$, if $x,\ y \ge 0\ and\ x^2+y^2 > 1$

32. $\sin^{-1}(-x) = -\sin^{-1}x,\ \cos^{-1}(-x) = x - \sin^{-1}x\ \tan^{-1}(-x) = -\tan^{-1}x,\ \cot^{-1}(-x) = x - \cot^{-1}x$

33. $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}2\sec^{-1}x = \tan^{-1}(2x^2-1)\ 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1-x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

34. $3\sin^{-1}x = \sin^{-1}(3x-4x^3),\ 3\cos^{-1}x = \cos^{-1}(4x^3-3x)\ 3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

35. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

36. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

37. $\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n = \tan^{-1}\left[\frac{S_1-S_2+S_3.....}{1-S_1+S_4-S_6+.....}\right]$ Where S_2 denotes the sum of the products of $x_1,\ x_2,......,x_n,$ taken k at a time.

Download Mathematics E-Books for C.B.S.E / I.C.S.E. / I.S.C. / JEE from www.mathstudy.in

Note : Important substitutions to simplify trigonometric expressions involving inverse trigonometric functions.

1. For $\sqrt{a^2-x^2}$; we substitute $x=a\sin\theta$ *or* $x=a\cos\theta$
2. For $\sqrt{a^2+x^2}$; we substitute $x=a\tan\theta$ *or* $x=a\cot\theta$
3. For $\sqrt{x^2-a^2}$; we substitute $x=a\sec\theta$ *or* $x=a\ \text{cosec}\theta$
4. For $\sqrt{\dfrac{a+x}{a-x}}$ *or* $\sqrt{\dfrac{a-x}{a+x}}$; we substitute $x=a\cos\theta$ *or* $x=a\cos 2\theta$

THINGS TO REMEMBER

1. Inverse Trigonometric Function : A function $f:A\to B$ is invertible if it is a bijection. The inverses of f is denoted by f^{-1} and is defined as $f^{-1}(y)=x \Leftrightarrow f(x)=y$. Clearly, domain of f^{-1} = range of f and range of f^{-1} = domain of f.
2. The inverse of sin function is defined as $\sin^{-1}x=\theta \Leftrightarrow \sin\theta=x$, where $\theta\in[-\pi/2,\ \pi/2]$ and $x\in[-1,\ 1]$.
3. Thus, $\sin^{-1}x$ has infinitely many values for given $x\in[-1,\ 1]$.
4. There is one value among these values which lies in the interval $[-\pi/2,\ \pi/2]$. This value is called the principal value.